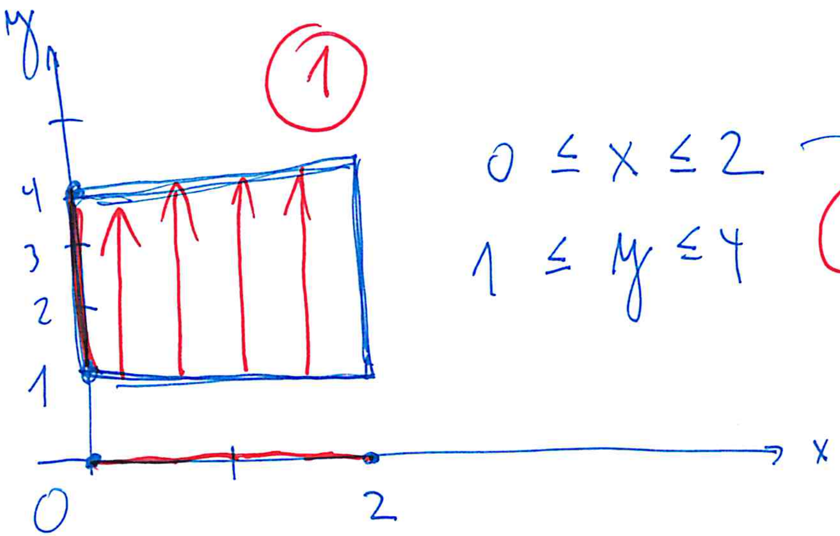
$$\iint_A (xy^2+y^3)\,dx\,dy = \int_0^2 \left[ \int_1^4 (xy^2+y^3)\,dy \right] dx = \int_0^2 \left(\frac{63}{3}x + \frac{255}{4}\right) dx =$$

2 4 *

A: obdélník $\langle 0,2 \rangle \times \langle 1,4 \rangle$

$$= \left[\frac{63}{6}x^2 + \frac{255}{4}x\right]_0^2 = \underbrace{\frac{126}{3}}_{=42} + \frac{255}{2} = \frac{339}{2}$$

1



$0 \le x \le 2$

$1 \le y \le 4$

3 *

$$\int_1^4 (xy^2+y^3)\,dy = \left[x\frac{y^3}{3} + \frac{y^4}{4}\right]_1^4 =$$

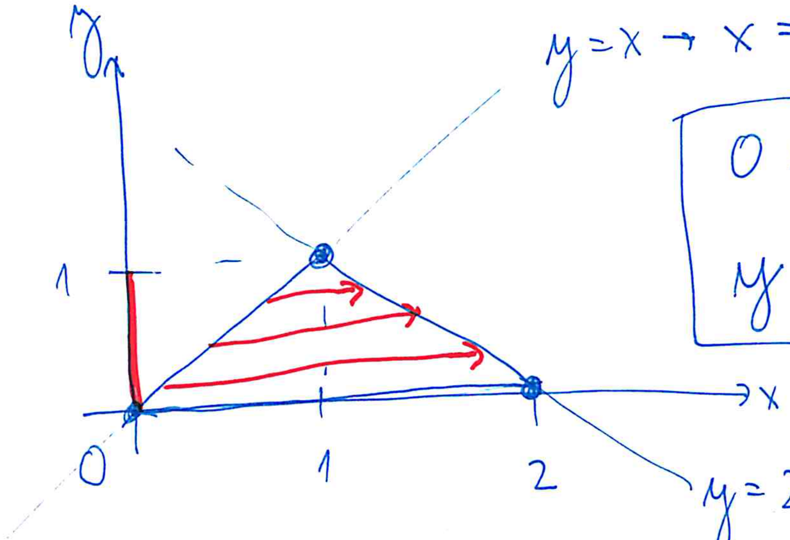
$$= \frac{64}{3}x + 64 - \left(\frac{1}{3}x + \frac{1}{4}\right) = \frac{63}{3}x + \frac{255}{4}$$

*

$$\iint_C (x-y)\,dx\,dy \overset{2}{=} \int_0^1 \left[\int_y^{2-y} (x-y)\,dx\right] dy = \int_0^1 (2y^2-4y+2)\,dy =$$

$$= \left[2\frac{y^3}{3} - 2y^2 + 2y\right]_0^1 = \frac{2}{3}$$

$C$ je $\Delta$ $[0,0], [1,1], [2,0]$




$y = x \to x = y$

$y = 2-x \to x = 2-y$

1

$$\boxed{\begin{aligned} 0 &\le y \le 1 \\ y &\le x \le 2-y \end{aligned}}$$

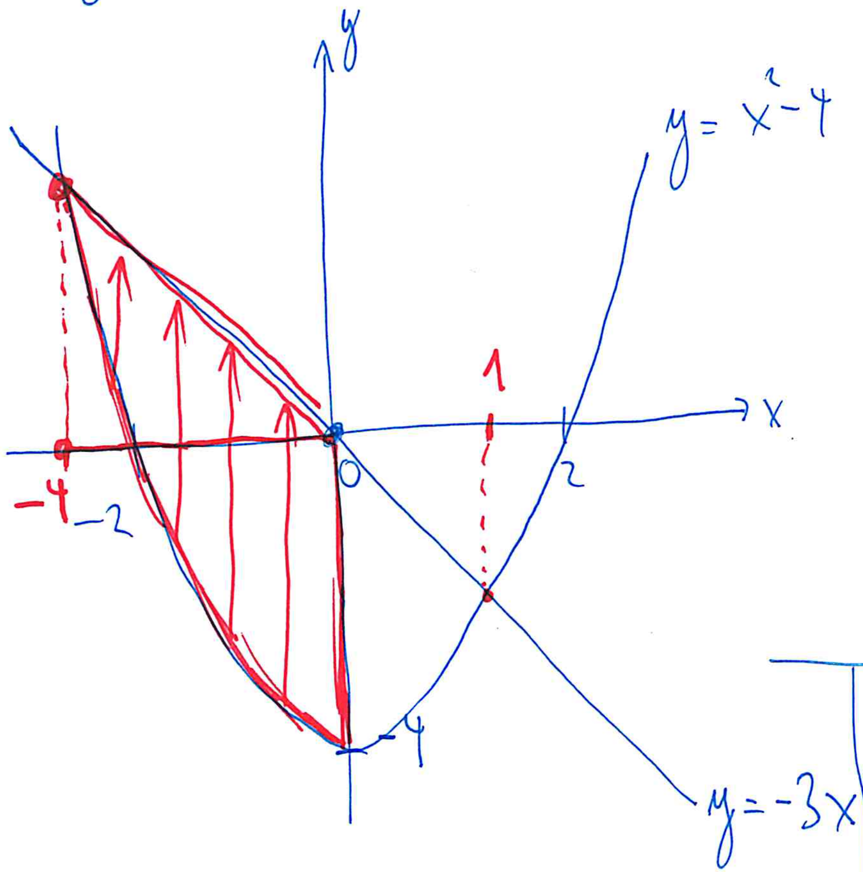
$*$ $$\int_y^{2-y} (x-y)\,dx = \left[\frac{1}{2}x^2 - yx\right]_y^{2-y} = \frac{1}{2}(2-y)^2 - y(2-y) - \left(\frac{1}{2}y^2 - y^2\right) =$$

$$= \frac{1}{2}(4-4y+y^2) - 2y + y^2 - \frac{1}{2}y^2 + y^2 = 2y^2 - 4y + 2$$

$$\iint_D (1+x)\cdot y\,dx\,dy = \int_{-4}^{0}\left[\int_{x^2-4}^{-3x}(1+x)\,y\,dy\right]dx = *$$

2

D je ohraničeno: $y=x^2-4$, $y=-3x$, $x\le 0$




průsečíky:

$x^2-4=-3x$

$x^2+3x-4=0$

$(x+4)(x-1)=0$

$x=-4$

$x=1$

$-4\le x\le 0$ 1

$x^2-4\le y\le -3x$

$$\int_{x^2-4}^{-3x} (1+x)\, y \, dy = (1+x)\frac{1}{2}\left[y^2\right]_{x^2-4}^{-3x} = (1+x)\frac{1}{2}\left(9x^2 - (x^2-4)^2\right) =$$

$$= \frac{1}{2}(1+x)\left(9x^2 - (x^4 - 8x^2 + 16)\right) = \frac{1}{2}(1+x)\left(-x^4 + 17x^2 - 16\right) =$$

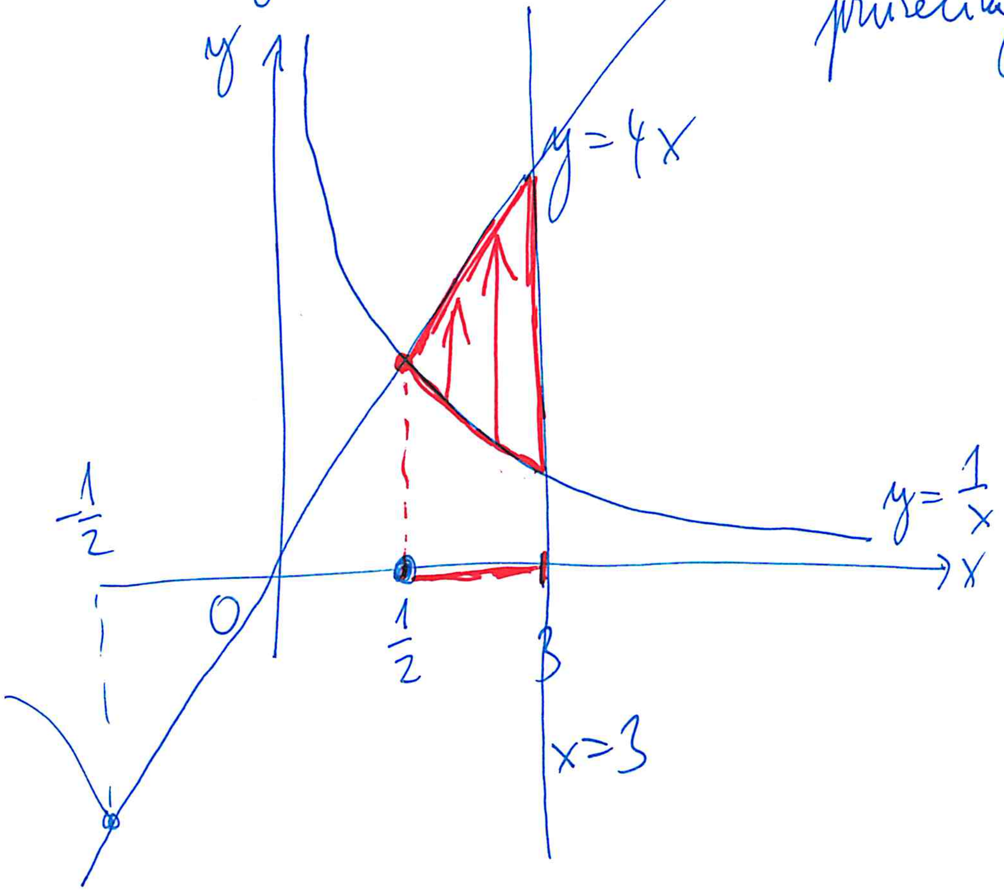
$$= \frac{1}{2}\left(-x^4 + 17x^2 - 16 - x^5 + 17x^3 - 16x\right)$$

$$\frac{1}{2}\int_{-4}^{0}\left(-x^5 - x^4 + 17x^3 + 17x^2 - 16x - 16\right)dx = \frac{1}{2}\left[-\frac{x^6}{6} - \frac{x^5}{5} + \frac{17x^4}{4} + 17\frac{x^3}{3} - 8x^2 - 16x\right]_{-4}^{0}$$

$$= \frac{1}{2}\left(0 - \left(-\frac{4^6}{6} - \frac{(-4)^5}{5} + \frac{17(-4)^4}{3} - 8(-4)^2 - 16(-4)\right)\right) = *$$

$$\iint_E \frac{x^2}{y^2}\,dx\,dy = \int_{\frac12}^{3}\left[\int_{\frac1x}^{4x}\frac{x^2}{y^2}\,dy\right]dx \overset{*}{=} \int_{\frac12}^{3}\left(x^3-\frac{x}{4}\right)dx =$$

$E:\ y=\frac{1}{x},\ y=4x,\ x=3$




průsečíky: $\frac{1}{x}=4x$

$\frac{1}{4}=x^2$

$x=\pm\frac{1}{2}$

$$\boxed{\begin{array}{l} \frac12 \le x \le 3 \\ \frac1x \le y \le 4x \end{array}}$$

$$= \left[\frac{x^4}{4}-\frac{x^2}{8}\right]_{\frac12}^{3} =$$

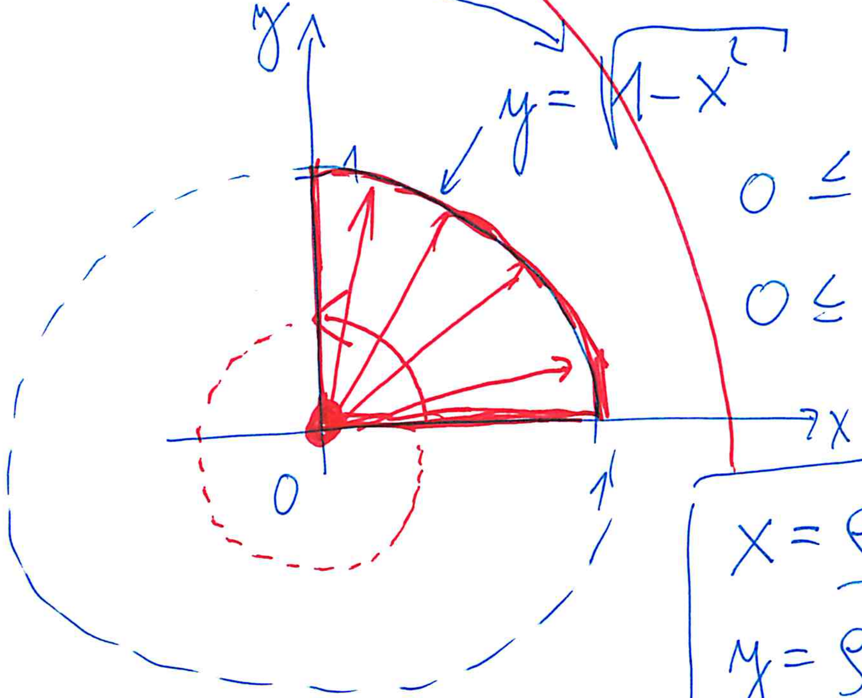
$$= \frac{81}{4}-\frac{9}{8}-\left(\frac{1}{64}-\frac{1}{32}\right) =$$

$$= \underline{\underline{\frac{1225}{64}}}$$

*)
$$\int_{\frac{1}{x}}^{4x} \frac{x^2}{y^2}\, dy = x^2 \left[\frac{y^{-1}}{-1}\right]_{\frac{1}{x}}^{4x} = -x^2 \left[\frac{1}{y}\right]_{\frac{1}{x}}^{4x} = -x^2\left(\frac{1}{4x} - x\right) = x^3 - \frac{x}{4}$$

$$\iint_F xy \, dx\,dy = \iint_F \underset{x}{\varrho\cos\varphi} \cdot \underset{y}{\varrho\sin\varphi} \cdot \underline{\varrho} \, d\varrho\, d\varphi = \iint_F \varrho^3 \cos\varphi \sin\varphi \, d\varrho \, d\varphi$$

$F: \ x^2 + y^2 \le 1, \ x \ge 0, \ y \ge 0$




$y = \sqrt{1-x^2}$

$0 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$

$x = \varrho\cos\varphi$

$y = \varrho\sin\varphi$

$0 \le \varphi \le \frac{\pi}{2}$

$0 \le \varrho \le 1$

$$\boxed{dx\,dy \longrightarrow \varrho \, d\varrho \, d\varphi}$$

$$\iint\limits_F \varrho^3 \cos\varphi \sin\varphi \, d\varrho \, d\varphi = \int_0^{\frac{\pi}{2}} \left( \int_0^1 \varrho^3 \cos\varphi \sin\varphi \, d\varrho \right) d\varphi \overset{*}{=} \frac{1}{8} \int_0^{\frac{\pi}{2}} \sin 2\varphi \, d\varphi =$$

$0 \le \varphi \le \frac{\pi}{2}$

$0 \le \varrho \le 1$

$$= \frac{1}{8} \cdot \frac{1}{2} \Big[ -\cos 2\varphi \Big]_0^{\frac{\pi}{2}} = \frac{1}{16} \big(1 - (-1)\big) = \frac{2}{16} = \frac{1}{8}$$

---

$\sin 2\varphi = 2 \sin\varphi \cos\varphi$

$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$

$$\int_0^1 \varrho^3 \cos\varphi \sin\varphi \, d\varrho = \cos\varphi \sin\varphi \left[ \frac{\varrho^4}{4} \right]_0^1 = \underline{\frac{1}{4} \cos\varphi \sin\varphi} = \frac{1}{8} \cdot 2 \cos\varphi \sin\varphi = \frac{1}{8} \sin 2\varphi$$

$*$ nebo

$$\int_0^{\frac{\pi}{2}} \frac{1}{4} \cos\varphi \sin\varphi \, d\varphi = \frac{1}{4} \int_0^1 t \, dt = \frac{1}{4} \left[ \frac{t^2}{2} \right]_0^1 = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

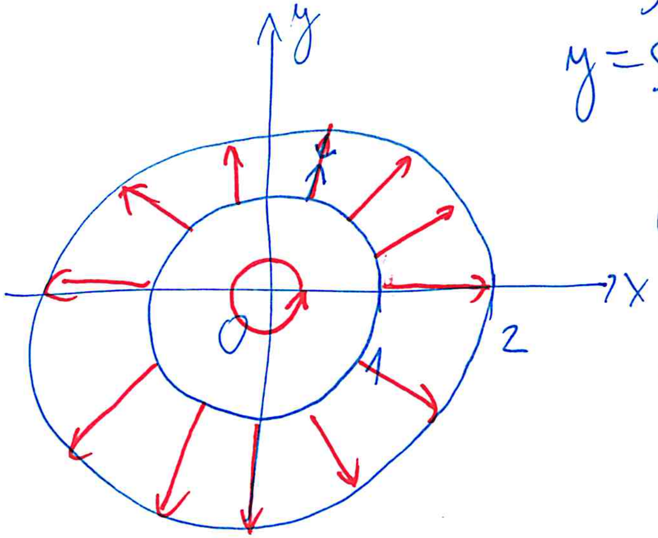
$t = \sin\varphi$

$dt = \cos\varphi \, d\varphi$

$$\iint_G \frac{1}{x^2+y^2}\,dx\,dy = \iint_G \frac{1}{\underbrace{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi}_{\rho^2(\underbrace{\cos^2\varphi+\sin^2\varphi}_{1})}}\,\rho\,d\rho\,d\varphi = \iint_G \frac{1}{\rho}\,d\rho\,d\varphi =$$

$G:\ 1 \le x^2+y^2 \le 4$

$x=\rho\cos\varphi$
$y=\rho\sin\varphi$




$0 \le \varphi \le 2\pi$

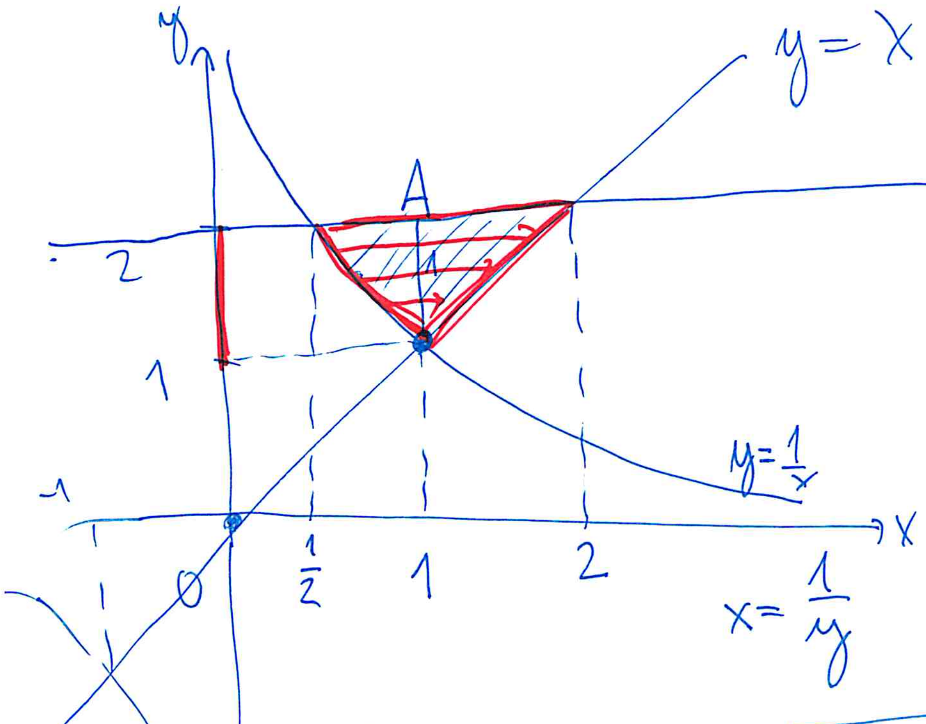
$1 \le \rho \le 2$

$$= \int_0^{2\pi}\left(\int_1^2 \frac{1}{\rho}\,d\rho\right)d\varphi =$$

$$= \int_0^{2\pi} [\ln\rho]_1^2\,d\varphi = \int_0^{2\pi} \ln 2\,d\varphi =$$

$$= \ln 2\,[\varphi]_0^{2\pi} = \underline{\underline{\ln 2 \cdot 2\pi}}$$

1) Obsah plochy $y=\frac{1}{x},\ y=x,\ y=2$




$$1 \le y \le 2$$

$$\frac{1}{y} \le x \le y$$

$$S_A = \iint_A 1\,dx\,dy =$$

$$= \int_1^2 \left( \int_{\frac{1}{y}}^{y} 1\,dx \right) dy =$$

$$= \int_1^2 \left[x\right]_{\frac{1}{y}}^{y} dy = \int_1^2 \left(y - \frac{1}{y}\right) dy =$$

$$= \left[\frac{y^2}{2} - \ln y\right]_1^2 = 2 - \ln 2 - \frac{1}{2} = \underline{\frac{3}{2} - \ln 2} \doteq 0{,}8$$

2) Obsah $x^2+y^2+x=0$, $x^2+y^2+4x=0$, $y=x$, $y=0$

$x^2+x+y^2=0$,

$(x+\frac{1}{2})^2-\frac{1}{4}+y^2=0$

$\boxed{(x+\frac{1}{2})^2+y^2=\frac{1}{4}}$ : $k_1$

$x^2+4x+y^2=0$

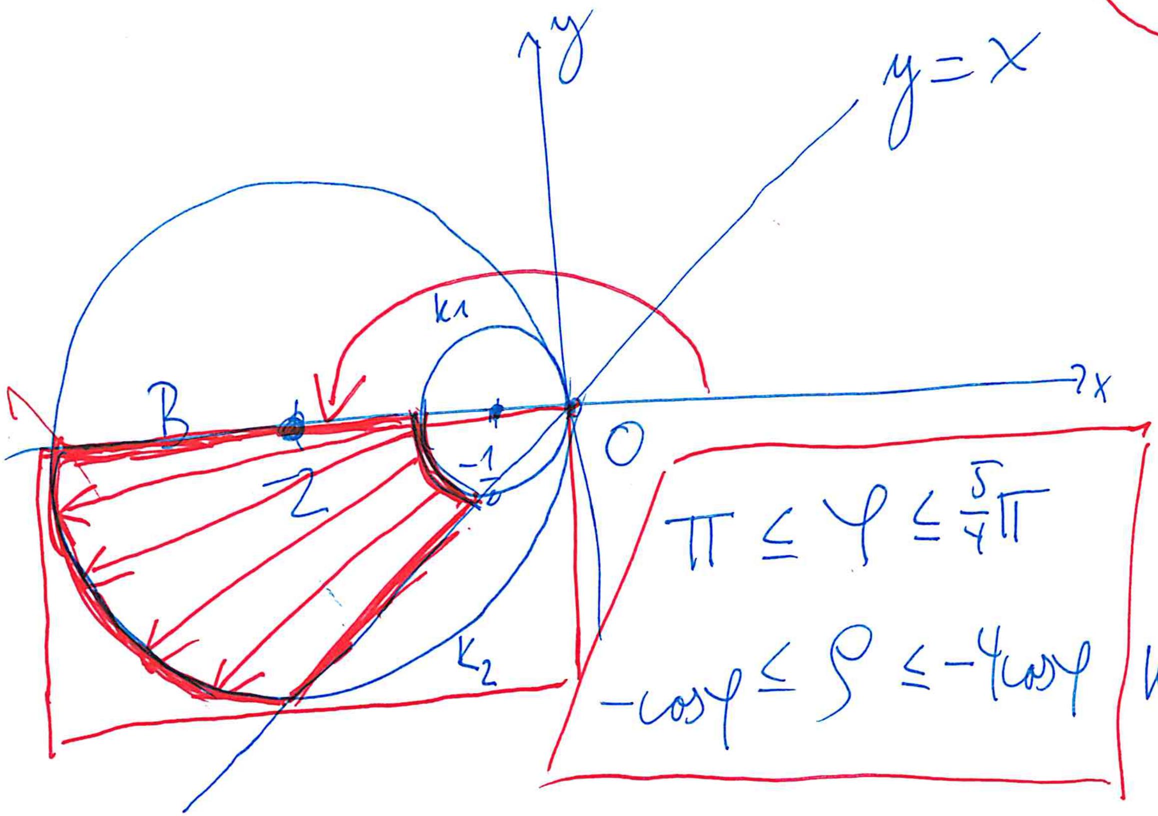
$(x+2)^2-4+y^2=0$

$\boxed{(x+2)^2+y^2=4}$ : $k_2$

$(x-x_S)^2+(y-y_S)^2=R^2$

$S_B=\iint_B 1\,dx\,dy$

$x=\rho\cos\varphi$
$y=\rho\sin\varphi$




$$\boxed{\begin{aligned}\pi\le\ &\varphi\le\tfrac{5}{4}\pi\\ -\cos\varphi\le\ &\rho\le-4\cos\varphi\end{aligned}}$$

$k_1: \rho^2\cos^2\varphi+\rho^2\sin^2\varphi+\rho\cos\varphi=0$

$\rho^2+\rho\cos\varphi=0$

$\rho(\rho+\cos\varphi)=0$ < $\rho=0$, $\boxed{\rho=-\cos\varphi}$

$k_2: \rho^2+4\rho\cos\varphi=0$

$\rho(\rho+4\cos\varphi)=0$ < $\rho=0$, $\boxed{\rho=-4\cos\varphi}$

$$S_B = \iint_B 1\,dx\,dy = \iint_B \rho\,d\rho\,d\varphi = \int_{\pi}^{\frac{5}{4}\pi} \left( \int_{-\cos\varphi}^{-4\cos\varphi} \rho\,d\rho \right) d\varphi = \int_{\pi}^{\frac{5}{4}\pi} \frac{15}{4}(\cos 2\varphi + 1)\,d\varphi \overset{*}{=}$$

$$\pi \le \varphi \le \frac{5}{4}\pi$$

$$-\cos\varphi \le \rho \le -4\cos\varphi$$

$$\overset{*}{=} \frac{15}{4}\left[\frac{1}{2}\cdot \sin(2\varphi) + \varphi\right]_{\pi}^{\frac{5}{4}\pi} = \frac{15}{4}\left(\frac{1}{2} + \frac{5}{4}\pi - \pi\right) =$$

$$\int_{-\cos\varphi}^{-4\cos\varphi} \rho\,d\rho = \left[\frac{1}{2}\rho^2\right]_{-\cos\varphi}^{-4\cos\varphi} = \frac{1}{2}\left(16\cos^2\varphi - \cos^2\varphi\right) = \underline{\frac{15}{2}\cos^2\varphi} = \frac{15}{4}(\cos 2\varphi + 1)$$

$$= \underline{\underline{\frac{15}{4}\left(\frac{1}{2} + \frac{1}{4}\pi\right)}}$$

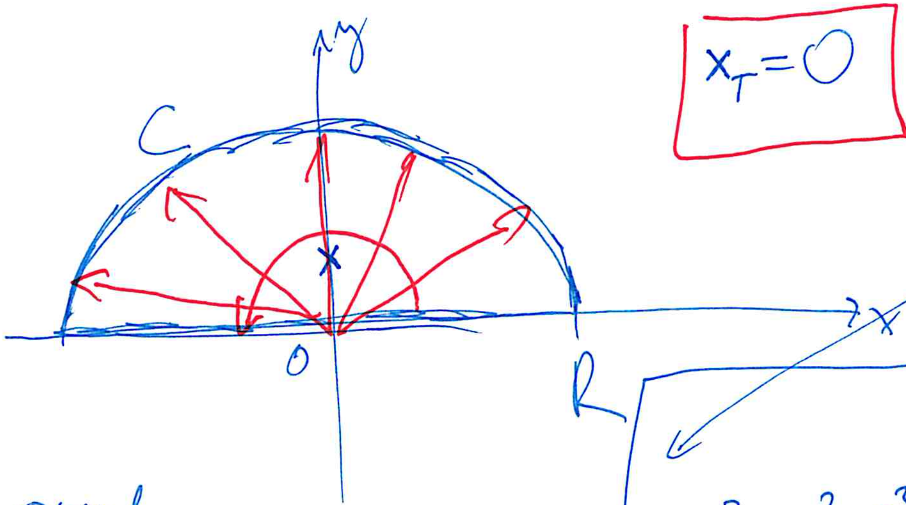
$$\doteq \underline{\underline{4{,}82}}$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = \underline{2\cos^2\varphi - 1} \rightarrow \cos^2\varphi = \frac{\cos 2\varphi + 1}{2}$$

$$\sin^2\varphi + \cos^2\varphi = 1 \rightarrow \sin^2\varphi = 1 - \cos^2\varphi$$

Těžiště

$h(x,y)=1 \qquad T=[x_T, y_T]$




$$\boxed{x_T=0}$$

$$y_T=\frac{1}{m}\iint_C y\cdot h(x,y)\,dx\,dy=\frac{2}{\pi R^2}\iint_C y\,dx\,dy$$

(1)

$$m=\iint_C h(x,y)\,dx\,dy=\iint_C 1\,dx\,dy=S\cdot h$$

$$\underline{m=\tfrac{1}{2}\pi R^2}$$

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$0\le\rho\le R$

$0\le\varphi\le\pi$

$$y_T=\frac{2}{\pi R^2}\cdot\frac{2}{3}R^3$$

$$\boxed{y_T=\frac{4R}{3\pi}=\frac{4}{3\pi}R}$$

$$\iint_C y\,dx\,dy=\iint_C \rho^2\sin\varphi\,d\rho\,d\varphi=\int_0^\pi\left[\int_0^R\rho^2\sin\varphi\,d\rho\right]d\varphi=\int_0^\pi\left[\frac{\rho^3}{3}\right]_0^R\sin\varphi\,d\varphi=$$

$$=\frac{R^3}{3}\int_0^\pi\sin\varphi\,d\varphi=\frac{R^3}{3}\left[-\cos\varphi\right]_0^\pi=\frac{R^3}{3}(1-(-1))=\frac{2}{3}R^3$$