VÝPOČET DVOJNÉHO INTEGRÁLU

PŘÍKLAD

Vypočtěte $\iint_A f(x,y)\,dx\,dy$, je-li:

a) $f(x,y) = x+y$, $A = \langle 0;1\rangle \times \langle 0;1\rangle$

$$\iint_A f(x,y)\,dx\,dy = \int_0^1 \int_0^1 (x+y)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} + xy\right]_0^1 dy = \int_0^1 \left(\frac{1}{2} + y\right) dy$$

$$= \left[\frac{1}{2}y + \frac{y^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

b) $f(x,y) = x \cdot \ln y$, $A = \langle 0;4\rangle \times \langle 1;e\rangle$

$$\iint_A f(x,y)\,dx\,dy = \int_1^e \int_0^4 x \cdot \ln y\,dx\,dy = \int_1^e \left[\frac{x^2}{2}\right]_0^4 \cdot \ln y\,dy =$$

$$= \int_1^e 8 \ln y\,dy = 8\int_1^e 1 \cdot \ln y\,dy = \begin{vmatrix} u' = 1 & v = \ln y \\ u = y & v' = \frac{1}{y} \end{vmatrix} =$$

$$= 8 \cdot \left([y \cdot \ln y]_1^e - \int_1^e y \cdot \frac{1}{y}\,dy \right) = 8 \cdot \left(e \cdot \ln e - 1 \cdot \ln 1 - \int_1^e dy \right) =$$

$$= 8\left(e - [y]_1^e\right) = 8(\cancel{e} - \cancel{e} + 1) = 8$$

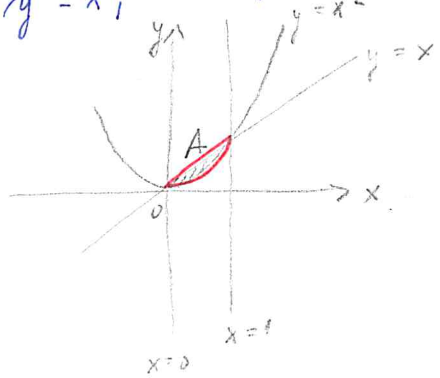
c) $f(x,y) = \frac{x}{y^2}$, $A = \langle 1;2\rangle \times \langle 3;4\rangle$

$$\iint_A f(x,y)\,dx\,dy = \int_3^4 \int_1^2 \frac{x}{y^2}\,dx\,dy = \int_3^4 \frac{1}{y^2}\left[\frac{x^2}{2}\right]_1^2 dy = \int_3^4 \frac{1}{2} \cdot \frac{1}{y^2} \cdot (4-1)\,dy =$$

$$= \int_3^4 \frac{3}{2} \cdot \frac{1}{y^2}\,dy = \frac{3}{2}\left[\frac{y^{-1}}{-1}\right]_3^4 = -\frac{3}{2}\left[\frac{1}{y}\right]_3^4 = -\frac{3}{2}\left(\frac{1}{4} - \frac{1}{3}\right) = -\frac{3}{2} \cdot \frac{-1}{12} =$$

$$= \frac{1}{8}$$

d) $f(x,y) = yx^2$, obor A je ohraničený křivkami $y = x^2$, $y = x$, $x = 0$, $x = 1$



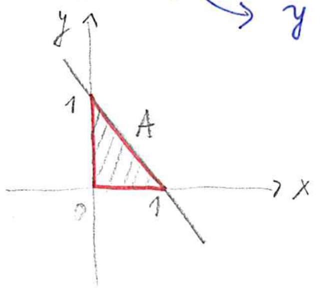
$A: \; 0 \le x \le 1$
$\quad x^2 \le y \le x$

$$\iint_A f(x,y)\,dx\,dy = \int_0^1\int_{x^2}^{x} y x^2\,dy\,dx = \int_0^1 x^2\left[\frac{y^2}{2}\right]_{x^2}^{x} dx =$$

$$= \int_0^1 \frac{1}{2}x^2\left(x^2 - x^4\right) dx = \frac{1}{2}\int\left(x^4 - x^6\right) dx = \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 =$$

$$= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{1}{2}\cdot\frac{2}{35} = \frac{1}{35}$$

e) $f(x,y) = x^2 + y^2$, obor A je ohraničený priamkami $x = 0$, $y = 0$, $x + y = 1 \rightarrow y = 1 - x$



$A: \; 0 \le x \le 1$
$ 0 \le y \le 1 - x$.

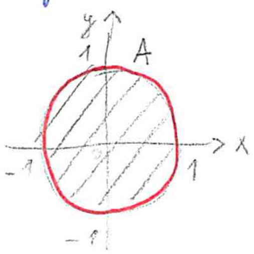
$$\iint_A f(x,y)\,dx\,dy = \int_0^1\int_0^{1-x}\left(x^2+y^2\right) dy\,dx = \int_0^1\left[x^2y + \frac{y^3}{3}\right]_0^{1-x} dx =$$

$$= \int_0^1\left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx = \int_0^1\left(x^2 - x^3 + \frac{1-3x+3x^2-x^3}{3}\right) dx =$$

$$= \int_0^1 \frac{-4x^3+6x^2-3x+1}{3}\,dx = \frac{1}{3}\left[-4\cdot\frac{x^4}{4} + 6\frac{x^3}{3} - 3\frac{x^2}{2} + x\right]_0^1 =$$

$$= \frac{1}{3}\left[-x^4 + 2x^3 - \frac{3}{2}x^2 + x\right]_0^1 = \frac{1}{3}\left(-1 + 2 - \frac{3}{2} + 1\right) = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

f) $f(x,y) = 1 - x^2 - y^2$, obor A je kruh $x^2 + y^2 \le 1$



polárne súradnice:

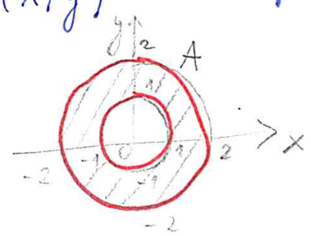
$A: \; 0 \le \varphi \le 2\pi$
$ 0 \le \rho \le 1$

$$\iint_A f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^1\left[1 - (\rho\cos\varphi)^2 - (\rho\sin\varphi)^2\right]\rho\,d\rho\,d\varphi =$$

$$= \int_0^{2\pi}\int_0^1\left(1-\rho^2\right)\rho\,d\rho\,d\varphi = \int_0^{2\pi}\int_0^1\left(\rho - \rho^3\right) d\rho\,d\varphi = \int_0^{2\pi}\left[\frac{\rho^2}{2} - \frac{\rho^4}{4}\right]_0^1 d\varphi =$$

$$= \int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{4}\right) d\varphi = \frac{1}{4}\left[\varphi\right]_0^{2\pi} = \frac{1}{4}\cdot 2\pi = \frac{\pi}{2}$$

g) $f(x,y) = x^2 + y^2$, obor A je medzikružie $1 \le x^2 + y^2 \le 4$



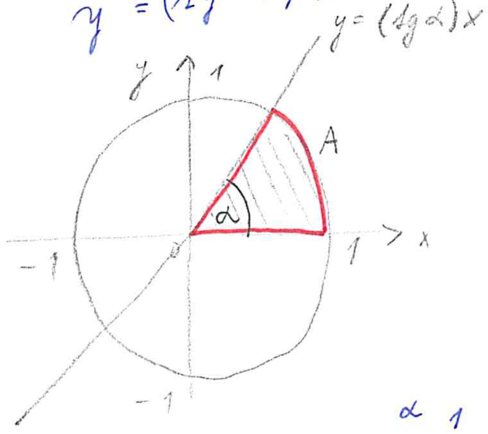
polárne súradnice:

$A: \; 0 \le \varphi \le 2\pi$
$ 1 \le \rho \le 2$

$$\iint_A f(x,y)\,dx\,dy = \int_0^{2\pi}\int_1^{2} \rho^2\cdot\rho\,d\rho\,d\varphi = \int_0^{2\pi}\left[\frac{\rho^4}{4}\right]_1^2 d\varphi = \int_0^{2\pi}\frac{1}{4}(16-1)\,d\varphi$$

$$= \frac{15}{4}\left[\varphi\right]_0^{2\pi} = \frac{15}{4}\cdot 2\pi = \frac{15\pi}{2}$$

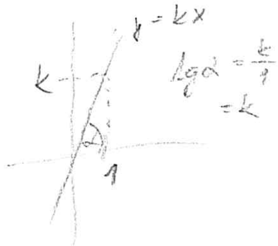
h) $f(x,y) = x^2+y^2$, obor A je kruhová výseč $y=0$, $x^2+y^2=1$, $y=(\operatorname{tg}\alpha)\,x$



polární souřadnice:

$A: \; 0 \le \varphi \le \alpha$

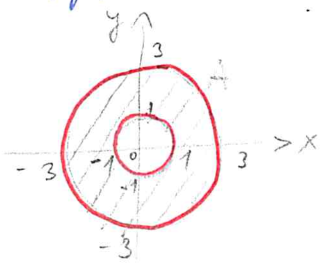
$\quad\; 0 \le \rho \le 1$



$$\iint_A f(x,y)\,dx\,dy = \int_0^{\alpha}\int_0^{1} \rho^2\cdot\rho\,d\rho\,d\varphi = \int_0^{\alpha}\left[\frac{\rho^4}{4}\right]_0^1 d\varphi = \int_0^{\alpha}\frac{1}{4}\,d\varphi = \frac{1}{4}\left[\varphi\right]_0^{\alpha}$$

$$= \frac{1}{4}\alpha$$

i) $f(x,y) = \sqrt{14-x^2-y^2}$, obor A je mezikruží $1 \le x^2+y^2 \le 9$



polární souřadnice:

$A: \; 0 \le \varphi \le 2\pi$

$\quad\; 1 \le \rho \le 3$

$$\iint_A f(x,y)\,dx\,dy = \int_0^{2\pi}\int_1^{3} \sqrt{14-\rho^2}\cdot\rho\,d\rho\,d\varphi = \left|\begin{array}{l} t = 14-\rho^2 \\ dt = -2\rho\,d\rho \\ d\rho = \dfrac{dt}{-2\rho} \\ \rho = 1 \to \rho = 13 \\ \rho = 3 \to \rho = 5 \end{array}\right| =$$

$$= \int_0^{2\pi}\left(\int_{13}^{5} t^{1/2}\cancel{\rho}\cdot\frac{-dt}{2\cancel{\rho}}\right)d\varphi = -\frac{1}{2}\int_0^{2\pi}\left(\int_{13}^{5} t^{1/2}\,dt\right)d\varphi =$$

$$= -\frac{1}{2}\int_0^{2\pi}\left[\frac{t^{3/2}}{3/2}\right]_{13}^{5} d\varphi = -\frac{1}{3}\int_0^{2\pi}\left(5\sqrt{5}-13\sqrt{13}\right)d\varphi = \frac{13\sqrt{13}-5\sqrt{5}}{3}\left[\varphi\right]_0^{2\pi} =$$

$$= \frac{13\sqrt{13}-5\sqrt{5}}{3}\cdot 2\pi = \frac{2}{3}\left(13\sqrt{13}-5\sqrt{5}\right)\pi$$

$$\int_0^{2\pi}\int_3^{?}\left(\sqrt{t}\cdot\rho\cdot\frac{dt}{-2\rho}\right)d\varphi$$