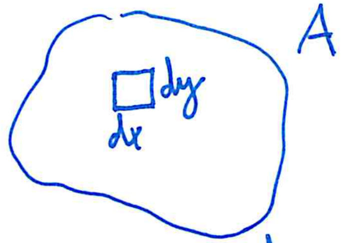
Použití dvojného integrálu

Obsah rovinného obrazce A

$$S(A) = \iint_A 1\,dx\,dy$$

Hmotnost r.o. A s plošnou hustotou $h = h(x,y)$

$$m(A) = \iint_A h(x,y)\,dx\,dy$$



$dS = dx\,dy$

$dm = h(x,y)\,dx\,dy$

Těžiště obrazce A s plošnou hustotou $h = h(x,y)$

$$x_T = \frac{1}{m(A)} \iint_A x\, h(x,y)\,dx\,dy \qquad \left(\text{SHB} \quad x_T = \frac{\sum x_i m_i}{m} \right)$$

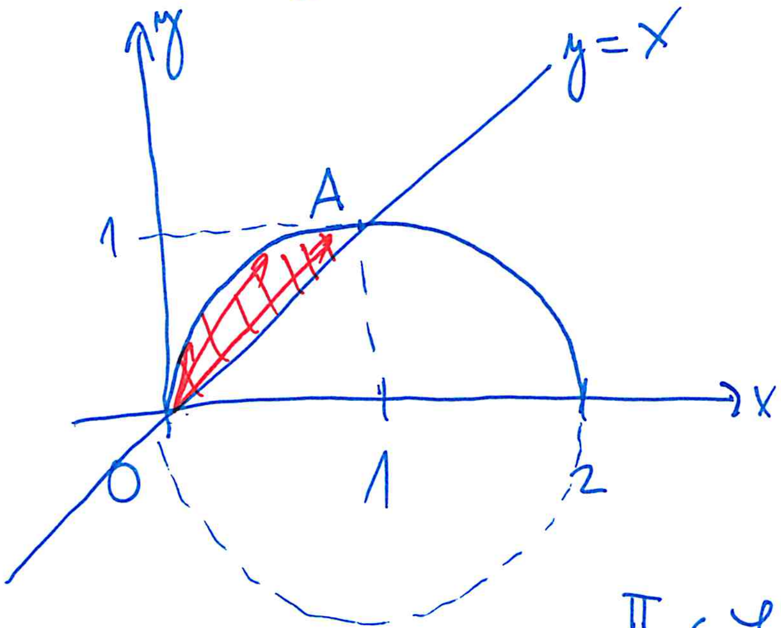
$$y_T = \frac{1}{m(A)} \iint_A y\, h(x,y)\,dx\,dy$$

Moment setrvačnosti A

vzhledem k ose x: $I_x = \iint_A y^2\, h(x,y)\,dx\,dy$

y: $I_y = \iint_A x^2\, h(x,y)\,dx\,dy$

Spočítejme obsah obrazce



$$S(A) = \iint_A 1 \, dx\,dy = \iint_A \varrho \, d\varrho \, d\varphi =$$

$x = \varrho \cos\varphi$

$y = \varrho \sin\varphi$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\varphi} \varrho \, d\varrho \right) d\varphi = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos 2\varphi) \, d\varphi =$$

$$= \left[\varphi + \tfrac{1}{2} \sin 2\varphi \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2} = \boxed{\frac{\pi}{4} - \frac{1}{2}}$$

$$\int_0^{2\cos\varphi} \varrho \, d\varrho = \frac{1}{2} \left[\varrho^2 \right]_0^{2\cos\varphi} = 2\cos^2\varphi = 1 + \cos 2\varphi$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = \cos^2\varphi - (1 - \cos^2\varphi) =$$
$$= 2\cos^2\varphi - 1$$

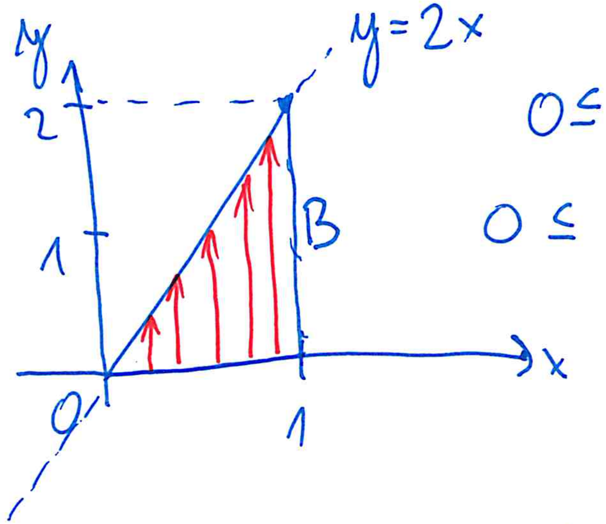
$$2\cos^2\varphi = \cos 2\varphi + 1$$

$$\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$$

$$0 \le \varrho \le 2\cos\varphi$$

$$(x-1)^2 + y^2 = 1$$
$$x^2 - 2x + y^2 = 0$$
$$\varrho^2 \cos^2\varphi + \varrho^2 \sin^2\varphi - 2\varrho\cos\varphi = 0$$
$$\varrho^2 - 2\varrho\cos\varphi = 0$$
$$\varrho \cdot (\varrho - 2\cos\varphi) = 0 \quad \begin{cases} \varrho = 2\cos\varphi \\ \varrho = 0 \end{cases}$$

Nalezněte těžiště trojúhelníku s vrcholy [0,0], [1,0], [1,2]



$h(x,y) = 1$

$0 \leq x \leq 1$

$0 \leq y \leq 2x$

$$m(B) = \iint_B h(x,y)\,dxdy = \iint_B 1\,dxdy = 1$$

$$x_T = \frac{1}{m}\iint_B x\,h(x,y)\,dxdy = \iint_B x\,dxdy = \int_0^1 \left(\int_0^{2x} x\,dy\right)dx = \int_0^1 x\,[y]_0^{2x}\,dx = \int_0^1 2x^2\,dx = \boxed{\frac{2}{3}}$$

$$y_T = \frac{1}{m}\iint_B y\,h(x,y)\,dxdy = \iint_B y\,dxdy = \int_0^1 \left(\int_0^{2x} y\,dy\right)dx = \int_0^1 \frac{1}{2}\left[y^2\right]_0^{2x}\,dx = \int_0^1 2x^2\,dx = \boxed{\frac{2}{3}}$$

$$T = \left[\frac{2}{3}, \frac{2}{3}\right]$$

Moment setrvačnosti ΔB vzhledem k ose x:

$$I_x = \iint y^2 h(x,y)\,dx\,dy = \iint y^2\,dx\,dy = \int_0^1 \left(\int_0^{2x} y^2\,dy \right) dx =$$

$$= \int_0^1 \left[\frac{y^3}{3} \right]_0^{2x} dx = \int_0^1 \frac{8}{3} x^3\,dx = \frac{8}{3} \left[\frac{x^4}{4} \right]_0^1 = \boxed{\frac{2}{3} = I_x}$$