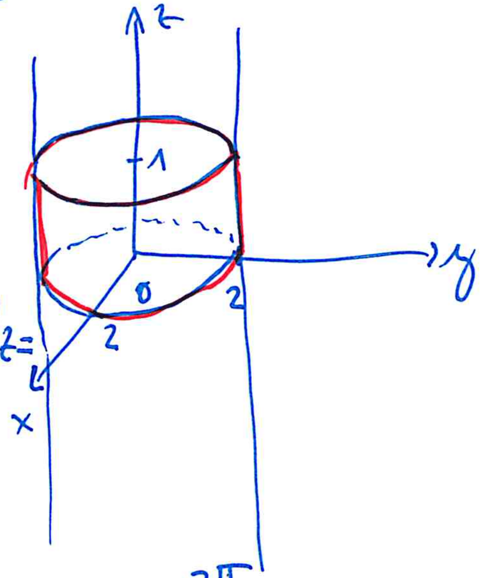
Výpočet T.I. ve válcových souřadnicích

Buď $u = f(x,y,z)$ spojitá na B. Pomocí transformačních rovnic $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$, $z = z$ dostaneme

$$u = f(\varrho\cos\varphi, \varrho\sin\varphi, z) = g(\varrho,\varphi,z)$$ a platí

$$\iiint\limits_{B} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{\bar{B}} g(\varrho,\varphi,z)\,\boxed{\varrho}\,d\varrho\,d\varphi\,dz$$

Príklady: $\iiint\limits_E z\cdot\sqrt{x^2+y^2}\,dx\,dy\,dz$, E je válec $x^2+y^2\le 4$, $0\le z\le 1$



$0\le\varrho\le 2$

$0\le\varphi\le 2\pi$

$0\le z\le 1$

$x=\varrho\cos\varphi$

$y=\varrho\sin\varphi$

$z=z$

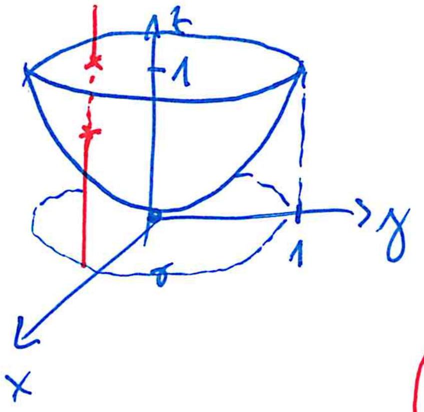
$$\iiint\limits_E z\sqrt{x^2+y^2}\,dx\,dy\,dz=\iiint\limits_E z\sqrt{\varrho^2\cos^2\varphi+\varrho^2\sin^2\varphi}\cdot\varrho\,d\varrho\,d\varphi\,dz=$$

$$=\iiint\limits_E z\varrho^2\,d\varrho\,d\varphi\,dz=\int_0^{2\pi}\left[\int_0^2\left(\int_0^1 z\varrho^2\,dz\right)d\varrho\right]d\varphi=\int_0^{2\pi}\frac{4}{3}\,d\varphi=\frac{4}{3}\big[\varphi\big]_0^{2\pi}=\underline{\underline{\frac{8\pi}{3}}}$$

$$\int_0^1 z\cdot\varrho^2\,dz=\frac{1}{2}\varrho^2\big[z^2\big]_0^1=\frac{1}{2}\varrho^2\;\longrightarrow\;\int_0^2\frac{1}{2}\varrho^2\,d\varrho=\frac{1}{6}\big[\varrho^3\big]_0^2=\frac{8}{6}=\frac{4}{3}$$

$$\iiint_F x^2 y \, dx\, dy\, dz$$

F: teleso ohraničené $z = x^2 + y^2$, $z = 1$



$0 \le \rho \le 1$

$0 \le \varphi \le 2\pi$

$\rho^2 \le z \le 1$

$x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $z = z$

$$\iiint_F x^2 y \, dx\,dy\,dz = \iiint_F \rho^2\cos^2\varphi \cdot \rho\sin\varphi \, \rho \, d\rho\, d\varphi\, dz = \iiint_F \rho^4\cos^2\varphi \sin\varphi \, d\rho\, d\varphi\, dz =$$

$$= \int_0^{2\pi} \left[\int_0^1 \left(\int_{\rho^2}^1 \rho^4 \cos^2\varphi \sin\varphi \, dz \right) d\rho \right] d\varphi \overset{3)}{=} \frac{2}{35}\int_0^{2\pi} \cos^2\varphi \sin\varphi \, d\varphi = -\frac{2}{35}\int_1^1 t^2\, dt = 0$$

$t = \cos\varphi$

$dt = -\sin\varphi \, d\varphi$

1) $$\int_{\rho^2}^1 \rho^4\cos^2\varphi \sin\varphi \, dz = \rho^4\cos^2\varphi\sin\varphi \left[z\right]_{\rho^2}^1 = \rho^4\cos^2\varphi \sin\varphi \,(1-\rho^2) = (\rho^4 - \rho^6)\cos^2\varphi\sin\varphi$$

2) $$\int_0^1 (\rho^4 - \rho^6)\cos^2\varphi \sin\varphi \, d\rho = \cos^2\varphi \sin\varphi \left[\frac{\rho^5}{5} - \frac{\rho^7}{7}\right]_0^1 = \cos^2\varphi\sin\varphi\left(\frac{1}{5} - \frac{1}{7}\right)$$

Výpočet T.I. ve sférických souřadnicích

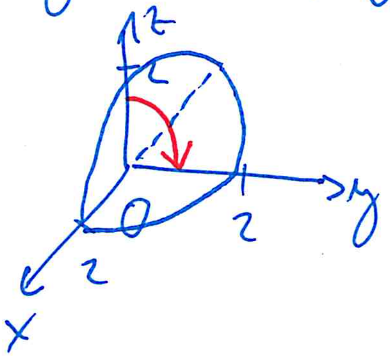
Buď $u = f(x,y,z)$ spojitá funkce na B. Pomocí transformace $x = r\cos\varphi \sin\vartheta$, $y = r \sin\varphi \sin\vartheta$, $z = r\cos\vartheta$ dostáváme

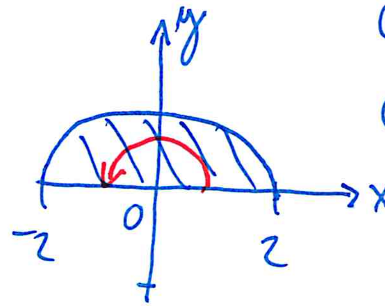
$$u = f(x,y,z) = f(r\cos\varphi \sin\vartheta,\ r\sin\varphi\sin\vartheta,\ r\cos\vartheta) = g(r,\varphi,\vartheta)$$

a platí

$$\iiint\limits_B f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{\overline{B}} g(r,\varphi,\vartheta)\, r^2 \sin\vartheta \, dr\, d\varphi\, d\vartheta$$

$$\iiint_G (x^2+y^2)\,dx\,dy\,dz \quad : G: x^2+y^2+z^2 \le 4,\ y \ge 0,\ z \ge 0$$





$0 \le \varphi \le \pi$

$0 \le r \le 2$

$0 \le \vartheta \le \frac{\pi}{2}$

$$\iiint_G (x^2+y^2)\,dx\,dy\,dz = \iiint_{\overline{G}} (r^2\cos^2\varphi\sin^2\vartheta + r^2\sin^2\varphi\sin^2\vartheta)\, r^2\sin\vartheta\, dr\,d\varphi\,d\vartheta =$$

$$= \iiint_{\overline{G}} r^2\sin^2\vartheta(\cos^2\varphi+\sin^2\varphi)\, r^2\sin\vartheta\, dr\,d\varphi\,d\vartheta = \iiint_{\overline{G}} r^4\sin^3\vartheta\, dr\,d\varphi\,d\vartheta =$$

$$= \int_0^{\pi}\left[\int_0^2\left(\int_0^{\frac{\pi}{2}} r^4\sin^3\vartheta\, d\vartheta\right) dr\right] d\varphi \overset{3)}{=} \int_0^{\pi}\frac{64}{15}\,d\varphi = \frac{64}{15}\left[\varphi\right]_0^{\pi} = \underline{\underline{\frac{64\pi}{15}}}$$

1) $\displaystyle\int_0^{\frac{\pi}{2}} r^4 \sin^3\vartheta \, d\vartheta = \int_0^{\frac{\pi}{2}} r^4 (1-\cos^2\vartheta)\sin\vartheta \, d\vartheta = -\int_1^0 r^4(1-t^2)\,dt =$

$t = \cos\varphi$

$-dt = \sin\varphi \, d\varphi$

$\displaystyle = r^4\left[t - \frac{t^3}{3}\right]_0^1 = r^4\left(1-\frac{1}{3}\right) = \underline{\underline{\frac{2}{3}r^4}}$

2) $\displaystyle\int_0^2 \frac{2}{3} r^4 \, dr = \frac{2}{15}\left[r^5\right]_0^2 = \underline{\underline{\frac{64}{15}}}$