APLIKACE TROJNÉHO INTEGRÁLU

těleso B s objemovou hustotou $h = h(x,y,z)$

$$V(B) = \iiint_B 1 \, dx\,dy\,dz \qquad \underset{\text{hmotnost}}{m(B)} = \iiint_B h(x,y,z)\, dx\,dy\,dz$$

těžiště $T = [x_T, y_T, z_T]$

$$x_T = \frac{1}{m} \iiint_B x \cdot h(x,y,z)\, dx\,dy\,dz$$

$$y_T = \frac{1}{m} \iiint_B y \cdot h(x,y,z)\, dx\,dy\,dz$$

$$z_T = \frac{1}{m} \iiint_B z \cdot h(x,y,z)\, dx\,dy\,dz$$

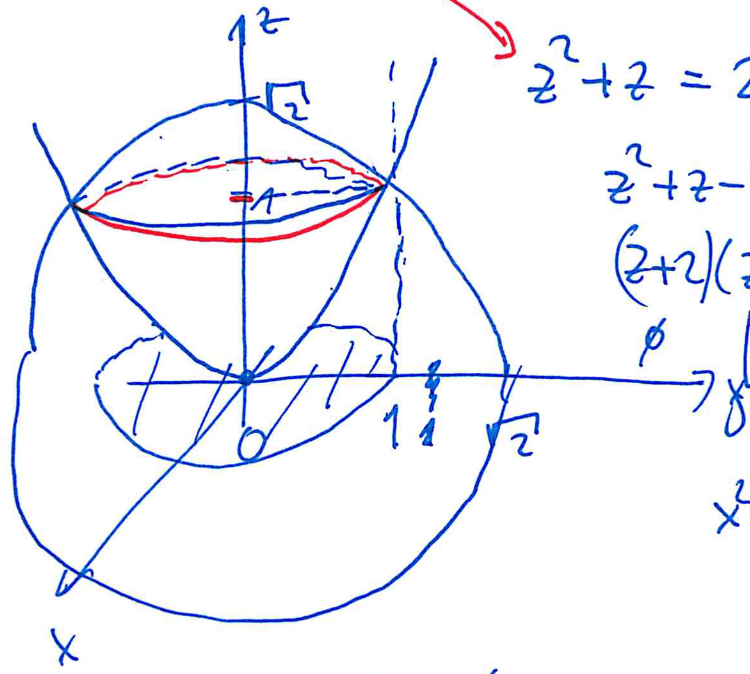
momenty setrvačnosti vzhledem k souřadným osám

$$I_x = \iiint_B (y^2+z^2) \cdot h(x,y,z)\, dx\,dy\,dz$$

$$I_y = \iiint_B (x^2+z^2)\, h(x,y,z)\, dx\,dy\,dz$$

$$I_z = \iiint_B (x^2+y^2) \cdot h(x,y,z)\, dx\,dy\,dz$$

Spočítejte objem tělesa B mezi $x^2+y^2+z^2=2$ a $z=x^2+y^2$ $(z\geq 0)$



$z^2+z=2$

$z^2+z-2=0$

$(z+2)(z-1)=0$

$\boxed{z=1}$

$x^2+y^2=z=1$

$$V=\iiint_B 1\,dx\,dy\,dz =$$

$$=\iiint_B \rho\,d\rho\,d\varphi\,dz =$$

$$=\int_0^{2\pi}\left[\int_0^1\left(\int_{\rho^2}^{\sqrt{2-\rho^2}}\rho\,dz\right)d\rho\right]d\varphi$$

VÁLCOVÉ SOUŘADNICE

$x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ z=z,\ J=\rho$

$0\leq\varphi\leq 2\pi$

$0\leq\rho\leq 1$

$\rho^2\leq z\leq\sqrt{2-\rho^2}$

(x^2+y^2) $(z=\sqrt{2-x^2-y^2})$

1) $\int_{\varrho^2}^{\sqrt{2-\varrho^2}} \varrho \, dz = \varrho \left[z \right]_{\varrho^2}^{\sqrt{2-\varrho^2}} = \varrho\sqrt{2-\varrho^2} - \varrho^3$

2) $\int_0^1 \varrho\sqrt{2-\varrho^2}\, d\varrho - \int_0^1 \varrho^3 d\varrho = \frac{1}{2}\int_0^1 \sqrt{2-t}\, dt - \left[\frac{\varrho^4}{4}\right]_0^1 = \frac{-1}{2}\left[\frac{(2-t)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 - \frac{1}{4} =$

$t = \varrho^2$

$dt = 2\varrho d\varrho$

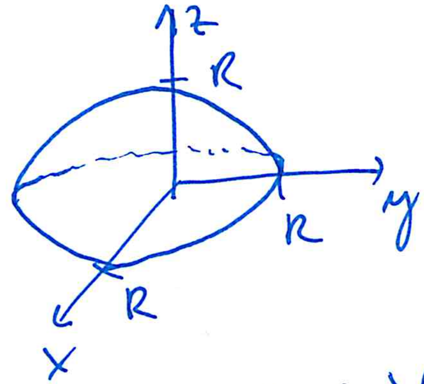
$\frac{1}{2} dt = \varrho d\varrho$

$= -\frac{1}{3}\left(1 - \sqrt{8}\right) - \frac{1}{4} = \frac{\sqrt{8}}{3} - \frac{7}{12}$

3) $V = \int_0^{2\pi} \left(\frac{\sqrt{8}}{3} - \frac{7}{12}\right) d\varphi = \left(\frac{\sqrt{8}}{3} - \frac{7}{12}\right) \cdot \left[\varphi\right]_0^{2\pi} = \underline{\underline{\left(\frac{\sqrt{8}}{3} - \frac{7}{12}\right) \cdot 2\pi}}$

Nalezněte souřadnice těžiště horní polokoule $C: x^2+y^2+z^2 \leq R^2,\ z \geq 0$

$$h(x,y,z) = 1$$



$$\boxed{\begin{array}{l} x_T = 0 \\ y_T = 0 \end{array}}$$

$$z_T = \frac{1}{m} \iiint\limits_C z \, dx\,dy\,dz = \frac{1}{\frac{2}{3}\pi R^3} \iiint\limits_C z \, dx\,dy\,dz$$

$$m = h \cdot V = 1 \cdot V = \frac{2}{3}\pi R^3$$

$$0 \leq \varphi \leq 2\pi$$

$$0 \leq r \leq R$$

$$0 \leq \vartheta \leq \frac{\pi}{2}$$

$$J = r^2 \sin\vartheta$$

$$z_T = \frac{3}{2\pi R^3} \iiint\limits_C z \, dx\,dy\,dz = \frac{3}{2\pi R^3} \iiint\limits_C \overbrace{r\cos\vartheta}^{z} \, \overbrace{r^2 \sin\vartheta}^{J} \, dr\,d\varphi\,d\vartheta$$

$$z_T = \frac{3}{2\pi R^3} \iiint\limits_C r^3 \sin\vartheta \cos\vartheta \, dr\,d\varphi\,d\vartheta = \frac{3}{2\pi R^3} \int_0^{2\pi} \left[\int_0^{\frac{\pi}{2}} \left(\int_0^R r^3 \sin\vartheta \cos\vartheta \, dr \right) d\vartheta \right] d\varphi$$

1) $$\int_0^R r^3 \sin\vartheta \cos\vartheta \, dr = \sin\vartheta\cos\vartheta \left[\frac{r^4}{4}\right]_0^R = \frac{1}{4}R^4 \sin\vartheta\cos\vartheta = \frac{1}{8}R^4 \sin 2\vartheta$$

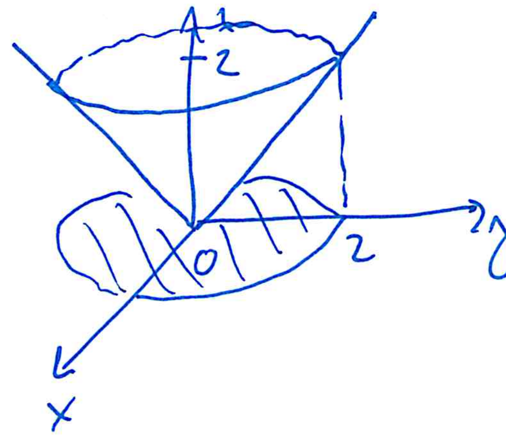
2) $$\int_0^{\frac{\pi}{2}} \frac{1}{8}R^4 \sin 2\vartheta \, d\vartheta = \frac{1}{8}R^4 \left[-\cos 2\vartheta\right]_0^{\frac{\pi}{2}} \cdot \frac{1}{2} = \frac{1}{16}R^4\left(1-(-1)\right) = \frac{1}{8}R^4$$

3) $$z_T = \frac{3}{2\pi R^3} \int_0^{2\pi} \frac{1}{8}R^4 \, d\varphi = \frac{3R}{16\pi}\left[\varphi\right]_0^{2\pi} = \frac{3R}{16\pi}\cdot 2\pi = \boxed{\frac{3R}{8} = z_T}$$

Moment setrvačnosti vzhledem k ose z tělesa ohraničeného $z=\sqrt{x^2+y^2}$

$z=2$ $\quad h(x,y,z)=1$

$z=\varrho$



$0\le\varrho\le 2$

$0\le\varphi\le 2\pi$

$\varrho\le z\le 2$

$x=\varrho\cos\varphi,\ y=\varrho\sin\varphi,\ z=z$

$$I_z=\iiint_D (x^2+y^2)\,dx\,dy\,dz=$$

$\varrho^2\cos^2\varphi+\varrho^2\sin^2\varphi=\varrho^2$

$$=\iiint_D \varrho^3\,d\varrho\,d\varphi\,dz=$$

$$=\int_0^{2\pi}\left[\int_0^2\left(\int_\varrho^2 \varrho^3\,dz\right)d\varrho\right]d\varphi \overset{3)}{=} \int_0^{2\pi}\frac{8}{5}\,d\varphi=\frac{8}{5}\left[\varphi\right]_0^{2\pi}=\boxed{\frac{16\pi}{5}}$$

1) $\displaystyle\int_\varrho^2 \varrho^3\,dz=\varrho^3\left[z\right]_\varrho^2=\varrho^3(2-\varrho)=2\varrho^3-\varrho^4$

2) $\displaystyle\int_0^2(2\varrho^3-\varrho^4)\,d\varrho=\left(\frac{\varrho^4}{2}-\frac{\varrho^5}{5}\right)_0^2=8-\frac{32}{5}=\underline{\underline{\frac{8}{5}}}$