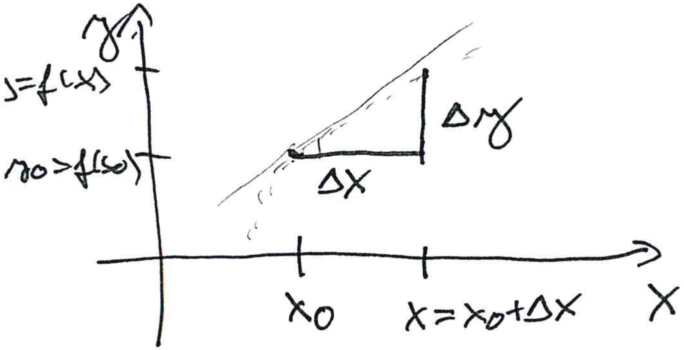


derivace funkce f v bodě x_0 je podíl nekonečně malých veličin $\frac{\Delta y}{\Delta x}$

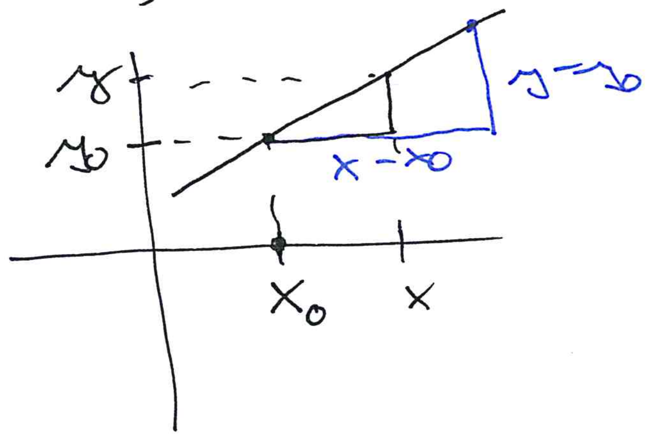
Příklad:

$f(x) = x^2$

$\Delta y = (x+\Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2$

$\frac{\Delta y}{\Delta x} = 2x + \Delta x = 2x$

geometrický význam



směrnice přímky: $y = kx + q$

$\frac{y-y_0}{x-x_0} = k$

$y - y_0 = k(x - x_0)$

rovnice tečny ke grafu funkce f v bodě x_0 $[x_0, f(x_0)]$

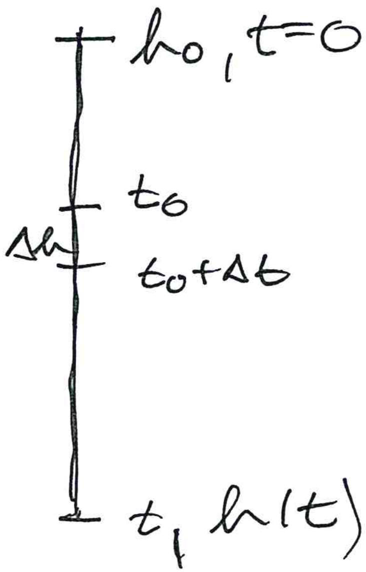
$y - f(x_0) = f'(x_0)(x - x_0)$

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0)}$$

Fyzikální význam:

$\cancel{s(t)}$

$$h(t) = h_0 - \frac{1}{2}gt^2$$

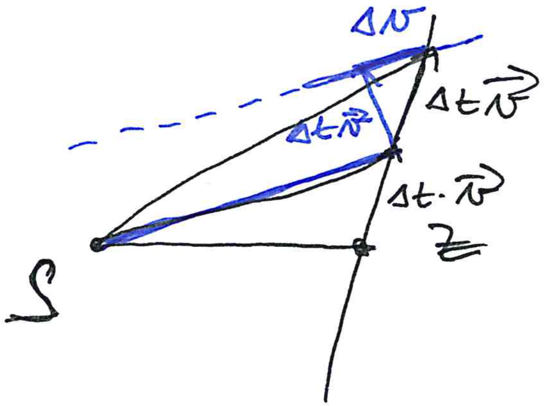


$$\frac{\Delta h}{\Delta t} = \bar{v}$$

$$v = h'(t) = 0 - \frac{1}{2}g\,2t = -gt$$

$$\bar{a} = \frac{\Delta v}{\Delta t}$$

$$a = v'(t) = -g$$



Definice:

Derivací funkce f v bodě x rozumíme (nazýváme) hodnotu limity

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{značíme } f'(x)$$

Derivací funkce f na intervalu $I = (a, b)$ rozumíme (nazýváme) funkci $x \mapsto f'(x)$.

Pro uzavřený interval $I = [a, b]$ je derivace f na I

$$x \mapsto \begin{cases} f'(x), & x \in (a, b) \\ f'_+(x), & x = a \\ f'_-(x), & x = b \end{cases}$$

$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$ derivace f v bodě x zprava

$$f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$ — " — zleva

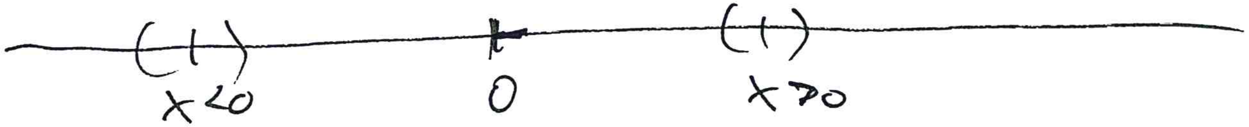
např.: ~~f~~ $f(x) = x^2$, $f'(x) = 2x$ na $\mathbb{R}$

Další příklady:

$$f(x) = x \cdot |x| = \begin{cases} x^2 & x > 0 \\ 0 & x = 0 \\ -x^2 & x < 0 \end{cases}$$

derivace je limita

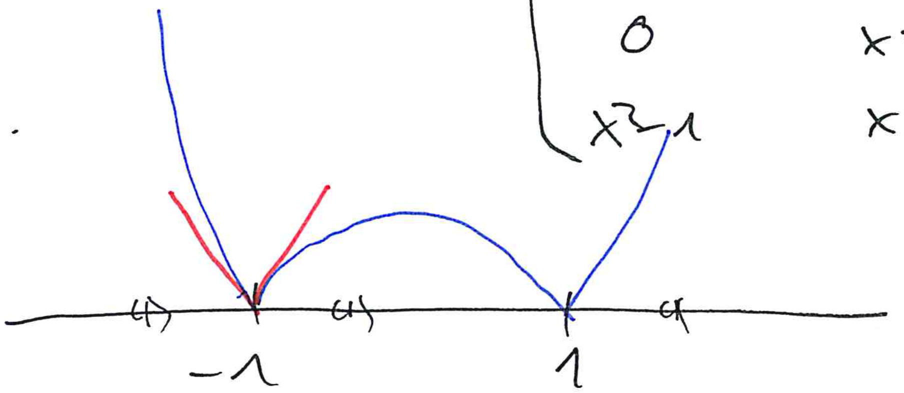
limita je lokální vlastnost – závisí na hodnotách v prstencovém okolí



$$f'(x) = \begin{cases} 2x & x > 0 \\ -2x & x < 0 \\ 0 & x = 0 \end{cases}$$

.. $\left.\begin{array}{l} f'_+(x) = 2x \\ f'_-(x) = -2x \end{array}\right\}$ pro $x=0$ se rovná

$$f(x) = |1-x^2| = \begin{cases} x^2-1 & x \in (-\infty,-1) \\ 0 & x=-1 \\ 1-x^2 & x \in (-1,1) \\ 0 & x = 1 \\ x^2-1 & x \in (1,+\infty) \end{cases}$$



$$f'(x) = \begin{cases} 2x & x \in (-\infty,-1) \\ \text{neex.} & x=-1 \\ -2x & x \in \cancel{(-\infty,-1)}\ (-1,1) \\ \text{neex.} & x=1 \\ 2x & x \in (1,+\infty) \end{cases}$$

$$f'_+(-1) = -2x\big|_{x=-1} = 2$$
$$f'_-(-1) = 2x\big|_{x=-1} = -2$$

$$(cf(x))' = c \cdot f'(x)$$
$$(f(x)+g(x))' = f'(x) + g'(x)$$

odvození:

$$(cf(x))' = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h\to 0} \frac{c\,(f(x+h)-f(x))}{h} = c\,\underbrace{\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)}_{=f'(x)} = c\,f'(x)$$

$$(f(x)+g(x))' = \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] = f'(x) + g'(x)$$

Další vzorce:

$$f(x) = x^n, \quad n \in \mathbb{N}$$

$$f(x+h) = (x+h)^n = x^n + n h x^{n-1} + \binom{n}{2} h^2 x^{n-2} + \dots$$

$$f(x) = x^3$$

$$f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3$$

$$\frac{f(x+h) - f(x)}{h} = 3x^2 + \cancel{3hx}\ 3xh + h^2$$

$$f'(x) = 3x^2$$

$$(x^n)' = n x^{n-1}, \quad n \in \mathbb{N}$$

$$n = 0: \quad (1)' = 0$$

$$(\sqrt{x})' = ? \qquad \frac{1}{2} x^{-\frac{1}{2}}$$

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{\sqrt{x+h}+\sqrt{x}} \xrightarrow[h \to 0]{} \frac{1}{2\sqrt{x}} =$$

$$= \frac{1}{2} x^{-\frac{1}{2}}$$

vzorec $(x^n)' = n x^{n-1}$ platí pro $n = \frac{1}{2}$

pro $n \in \mathbb{R}$

$$(f(x)\cdot g(x))' \overset{?}{=} f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = ?$$

$$\lim_{h\to 0} \frac{f(x+a)\cdot g(x+a) - f(x)\cdot g(x)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+a)\left[g(x+a)-g(x)\right] + \overbrace{f(x+a)\,g(x) - f(x)\cdot g(x)}^{g(x)\left[f(x+a)-f(x)\right]}}{h} =$$

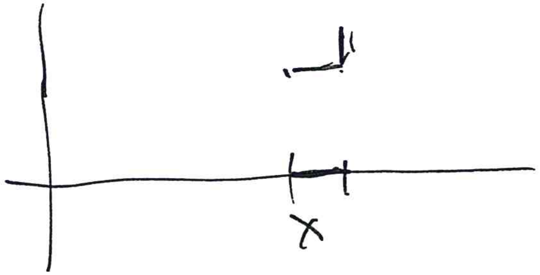
$$= \lim_{h\to 0} \left[f(x+a)\,\frac{g(x+a)-g(x)}{h} + g(x)\,\frac{f(x+a)-f(x)}{h} \right] =$$

(věta o limitě součtu a součinu

$$= \underbrace{\lim_{h\to 0} f(x+a)}_{=f(x)\ \ldots\ \text{platí, je-li } f \text{ spojitá v bodě } x} \cdot g'(x) + g(x)\cdot f'(x)$$

Věta:

Má-li funkce f v bodě x derivaci $f'(x) \in \mathbb{R}$ (tj. je konečná), pak je f spojitá v bodě x.



Důkaz:

$$f(x+h)-f(x) = \underbrace{\frac{f(x+h)-f(x)}{h}}_{\to f'(x)} \cdot \underbrace{h}_{\to 0} \xrightarrow{h\to 0} f'(x)\cdot 0 = 0$$

$$\lim_{h\to 0}\left(f(x+h)-f(x)\right) = 0$$

$$\lim_{h\to 0} f(\cancel{x+}h) = f(x)$$ .. f je spojitá v bodě x.

Poznámka: platí tedy implikace $f'(x)$ ex., je konečná $\Rightarrow$ f je spojitá v x

obrácená neplatí $\not\Leftarrow$

$$\left(\frac{f(x)}{g(x)}\right)' \overset{?}{=} \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

$$\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \frac{f(x+h)g(x) - f(x)\,g(x+h)}{\left(g(x+h)\cdot g(x)\right)h} =$$

$\to f'(x)$ for $h \to 0$

$f(x)\left(g(x) - g(x+h)\right)$

$$= \frac{\left(f(x+h) - f(x)\right)\cdot g(x) + f(x)\cdot g(x) - f(x)\,g(x+h)}{g(x+h)\cdot g(x)\cdot h} =$$

$\to -g'(x)$

for $h \to 0$

$$= \frac{f(x+h)-f(x)}{h}\cdot\frac{g(x)}{g(x+h)\cdot g(x)} + \frac{g(x)-g(x+h)}{h}\cdot\frac{f(x)}{g(x)\cdot g(x+h)} \longrightarrow \frac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{(g(x))^2}$$

$\to g(x)$ $\qquad$ $\to g(x)$

Příklady!

$$\left(\frac{\sqrt{x}-x\sqrt{x+1}}{x^2+1}\right)' = \frac{\left(\frac{1}{2}x^{-\frac{1}{2}}-\sqrt{x+1}-\frac{1}{2}x(x+1)^{-\frac{1}{2}}\right)(x^2+1)-(\sqrt{x}-x\sqrt{x+1})\cdot 2x}{(x^2+1)^2}$$

$$-x\sqrt{x+1} = (-x)'\sqrt{x+1} + (-x)\left(\sqrt{x+1}\right)' =$$

$$= -1\sqrt{x+1} - x\cdot\frac{1}{2}(x+1)^{-\frac{1}{2}}\cdot 1$$

$$(x^1)' = 1\cdot x^0 = 1$$

derivace složené fukce: $\left(\sqrt{g(x)}\right)' = \frac{1}{2}\left(g(x)^{\frac{1}{2}}\right)' =$

$$= \frac{1}{2}\left(g(x)\right)^{-\frac{1}{2}}\cdot g'(x)$$

Derivace složené funkce:

$$\Big(f(g(x_0))\Big)'$$

$$\frac{f(g(x_0+h)) - f(g(x_0))}{h} = \frac{f(g(x_0+h)) - f(g(x_0))}{g(x_0+h) - g(x_0)} \cdot \frac{g(x_0+h) - g(x_0)}{h}$$

$y_0 = g(x_0)$ $\quad$ $x = x_0 + h$

$z_0 = f(y_0)$ $\quad$ $\cancel{y = f(x) = f(x_0+h)}$

$\cancel{z = g(y) =}$

$y = g(x) = g(x_0+h)$

$z = f(y) = f(g(x_0+h))$

$$\frac{z-z_0}{x-x_0} = \frac{z-z_0}{y-y_0} \cdot \frac{y-y_0}{x-x_0} \to g'(x_0)$$

$$f'(g(x_0))$$

$$f'(y_0)$$

$$\left(f(g(x))\right)'\Big|_{x=x_0} = f'(y_0) \cdot g'(x_0)$$

$$\left(\sqrt{\underbrace{x^5+3x}_{y}}\right)' = \frac{1}{2}\left(x^5+3x\right)^{-\frac{1}{2}} \cdot \left(5x^4+3\right)$$

$$\left(\left(\sqrt{x}+1\right)^2\right)' =$$

$$= 2\left(\sqrt{x}+1\right) \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

$$\left(x+2\sqrt{x}+1\right)' =$$

$$= 1 + 2 \cdot \frac{1}{2}x^{-\frac{1}{2}} = 1+x^{-\frac{1}{2}}$$