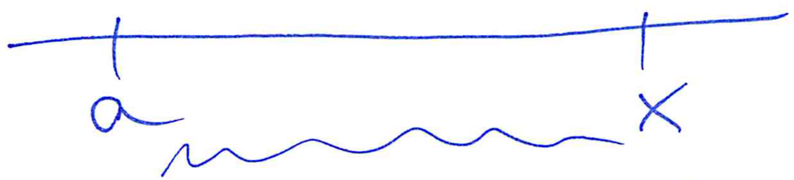
Lagrangeův tvar zbytku Taylorova polynomu

$$R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x-a)$$



c je někde mezi a, x :

$$\Big(\exists c \in (a,x) \cup (x,a)\Big)\left(R_1(x) = \frac{1}{2} f''(c)(x-a)^2\right)$$

Příklad:

$f(x) = \sqrt{x}, \quad a = 1,$

$T_1(x) = 1 + \frac{1}{2}(x-1)$

$f(1.2) \doteq 1.0954$ $\qquad$ $f(0.9) = 0.9487$

$T_1(1.2) = 1.1$ $\qquad$ $T_1(0.9) = 0.95$

$R_1(1.2) = -0.0046$ $\qquad$ $R_1(0.9) = -0.0013$

odhad R_1:

$f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$

$\frac{1}{2} f''(x) = -\frac{1}{8} x^{-\frac{3}{2}}$

$\frac{1}{2} f''(c) = -\frac{1}{8} c^{-\frac{3}{2}}$

$x = 1.2$

$c \in [1, 1.2] = I$

Obraz I ve funkci $c \mapsto -\frac{1}{8} c^{-\frac{3}{2}} = \frac{-1}{8\sqrt{c^3}}$

$\frac{1}{2} f''(c) \in \left[-\frac{1}{8}, \frac{-1}{8\sqrt{1.2^3}}\right]$ obraz monotonní funkce

$\left|\frac{1}{2} f''(c)\right| \le \frac{1}{8}$ zajímá nás horní odhad
tj. nejhorší případ

$|R_1(x)| \le \frac{1}{8}(x-1)^2$, $R_1(x) < 0$

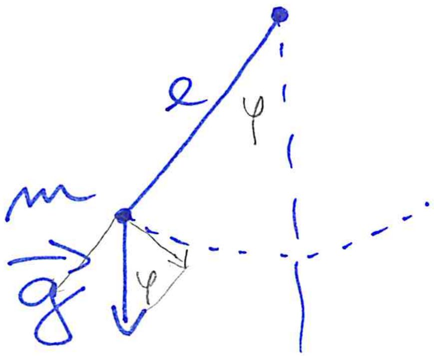
$x = 1.2$ $|R_1(1.2)| \le \frac{1}{8} 0,2^2 = \frac{0.04}{8} = 0.005$

$x = 0.9$ $|R_1(0.9)| \le \cancel{\frac{1}{8}(-0.1)^2 = \frac{0.01}{8} = 0.00125}$

$\boxed{\text{horní hranice je } \frac{1}{8\sqrt{0.9^3}}(-0.1)^2}$

$\boxed{0.00125 < 0.0013}$ to je divné

Příklad :



$$m l \varphi'' = m g \sin \varphi$$

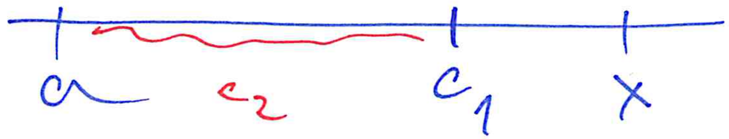
malé φ : $\sin \varphi \doteq \varphi$

$$l \varphi'' = g \varphi$$

Odvození Lagrangeova tvaru zbytku Taylorova polynomu

$$F(t) = R_1(t) - \frac{R_1(x)}{(x-a)^2}(t-a)^2$$

minule: $F(a) = F(x) = 0$



$$F'(t) = R_1'(t) - \frac{R_1(x)}{(x-a)^2} 2(t-a)$$

Rolleova věta: $\big(\exists c_1 \in (a,x) \cup (x,a)\big)\big(\boxed{F'(c_1) = 0}\big)$

zároveň: $F'(a) = \underbrace{R_1'(a)}_{=0} - \underbrace{\frac{R_1(x)}{(x-a)^2} 2\underbrace{(a-a)}_{=0}}_{=0}$

$$R_1'(t) = f'(t) - 0 - f'(a)(1-0)$$

proto: $\boxed{F'(a) = 0}$

F' splňuje předpoklady Rolleovy věty na $[a, c_1]$ $([c_1, a])$

$$\left(F'(t)\right)' = F''(t) = \underbrace{R_1''(t)}_{= f''(t) - 0 - 0} - \frac{R_1(x)}{(x-a)^2}\cdot 2(1-0)$$

Za předpokladu, že f má druhou derivaci na intervalu $[a,x]$ (případně $[x,a]$), tak můžeme použít ~~Ro~~ Rolleovu větu na funkci F':

tedy $(\exists c_2)(\cancel{F''(c)}\ F''(c_2) = 0)$

$$F''(c_2) = f''(c_2) - \frac{\boxed{R_1(x)}}{(x-a)^2}\cdot 2 = 0$$

$$\boxed{R_1(x) = \frac{1}{2} f''(c_2)(x-a)^2}$$

Lagrangeův tvar zbytku Taylorova polynomu

$f(x) = \sqrt{x}$, $a = 1$ $\qquad$ $f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$

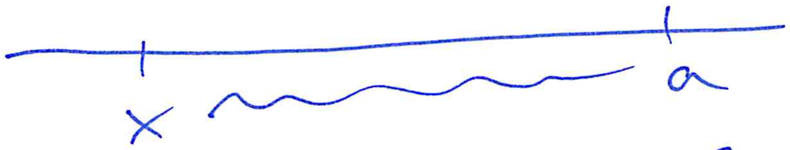
$T_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$

Lagrangeův tvar zbytku Taylorova polynomu 2. stupně

$$R_2(x) = f(x) - T_2(x)$$

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

$$R_2(x) = f(x) - f(a) - f'(a)(x-a) - \frac{1}{2} f''(a)(x-a)^2$$



c je mezi x, a ;

$$\left(\exists c \in (a,x) \cup (x,a)\right)\left(R_2(x) = \frac{1}{3!} f'''(c)(x-a)^3\right)$$