$Q_n(x) = c_n (1-x^2)^n$, c_n je zvoleno tak, že $\int_{-1}^{1} Q_n(x)\,dx = 1$

Dosazením dostaneme:

$$\int_{-1}^{1} c_n (1-x^2)^n \, dx = 1$$

$$c_n \int_{-1}^{1} (1-x^2)^n \, dx = 1$$

$$c_n = \frac{1}{\int_{-1}^{1} (1-x^2)^n \, dx}$$

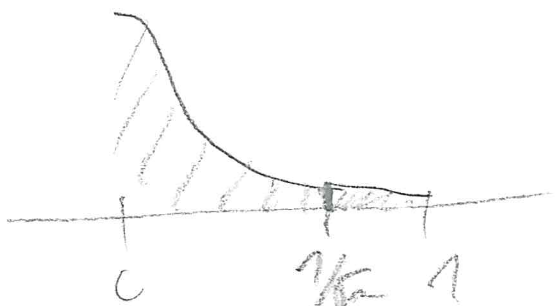
Spočítáme horní odhad – dokážeme, že $(\forall n \in \mathbb{N})(0 < c_n < \sqrt{n})$

$$\int_{-1}^{1}(1-x^2)^n\,dx = 2\int_{0}^{1}(1-x^2)^n\,dx$$

$$\int_{-1}^{0}(1-x^2)^n\,dx = \left[\begin{matrix} y=-x \\ dy=-dx \end{matrix}\right] = \int_{1}^{0}-(1-y^2)^n\,dy = \int_{0}^{1}(1-y^2)^n\,dy$$

$1/\sqrt{n} < 1$

$$2\int_{0}^{1}(1-x^2)^n\,dx > 2\int_{0}^{1/\sqrt{n}}(1-x^2)^n\,dx$$



$$\int_{0}^{1}(\ )\,dx = \int_{0}^{1/\sqrt{n}}(\ )\,dx + \underbrace{\int_{1/\sqrt{n}}^{1}\overbrace{(\ )}^{>0}\,dx}_{>0}$$

Bernoulliova nerovnost:

$$(1-x^2)^n \geq 1-nx^2$$

$$(1+y)^n \geq 1+ny, \quad y > -2 \quad (\text{dělme} \ \cancel{\text{nadále}} \ \text{jen pro } y > -1)$$

Tedy pro $x \in [-1,1]$ je $(1-x^2)^n \geq 1 - nx^2$

Vezmeme pro x, pro něž je $1 - nx^2 \geq 0$,

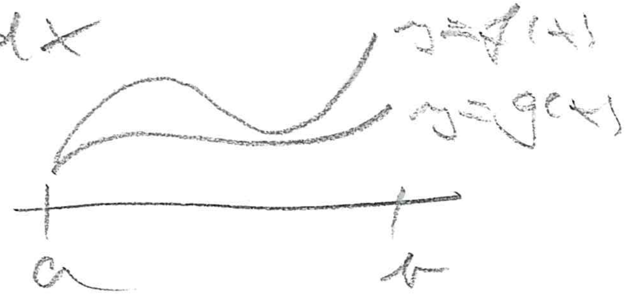
dostaneme: $1 \geq nx^2$, $x^2 \leq \frac{1}{n}$ odkud pro $x \geq 0$ je $x < \frac{1}{\sqrt{n}}$

Dostáváme:

$$2\int_0^{1/\sqrt{n}} (1-x^2)^n \, dx \geq 2 \int_0^{1/\sqrt{n}} 1 - nx^2 \, dx$$

Použití: Je-li $(\forall x \in [a,b])(g(x) \leq f(x))$, pak je

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$$



Spočítáme:

$$2\int_0^{1/\sqrt{n}} 1-nx^2\,dx = 2\left[x-\frac{nx^3}{3}\right]_0^{1/\sqrt{n}} =$$

$$= 2\left(\frac{1}{\sqrt{n}}-\frac{n}{3(\sqrt{n})^3}\right)-0 = 2\left(\frac{1}{\sqrt{n}}-\frac{1}{3\sqrt{n}}\right) =$$

$$= 2\,\frac{3-1}{3\sqrt{n}} = \frac{4\cancel{\sqrt{n}}}{3\sqrt{n}} > \cancel{\sqrt{n}}\ \frac{1}{\sqrt{n}}$$

Dostali jsme:

$$\int_{-1}^{1}(1-x^2)^n\,dx > \frac{1}{\sqrt{n}}, \text{ a tedy } c_n < \cancel{\frac{1}{\sqrt{n}}}\sqrt{n}$$

Budeme uvažovat f spojitou na $[0,1]$.

Označme:

$$P_n(x) = \int_{-1}^{1} Q_n(t)\cdot f(x+t)\,dt$$

$x+t=1 \;\dots\; t=1-x$

rozšíříme

zde (definujeme) $f(x)=0$

$x+t=0$

$t=-x$

$f(x)=0$

$x\in[0,1]$

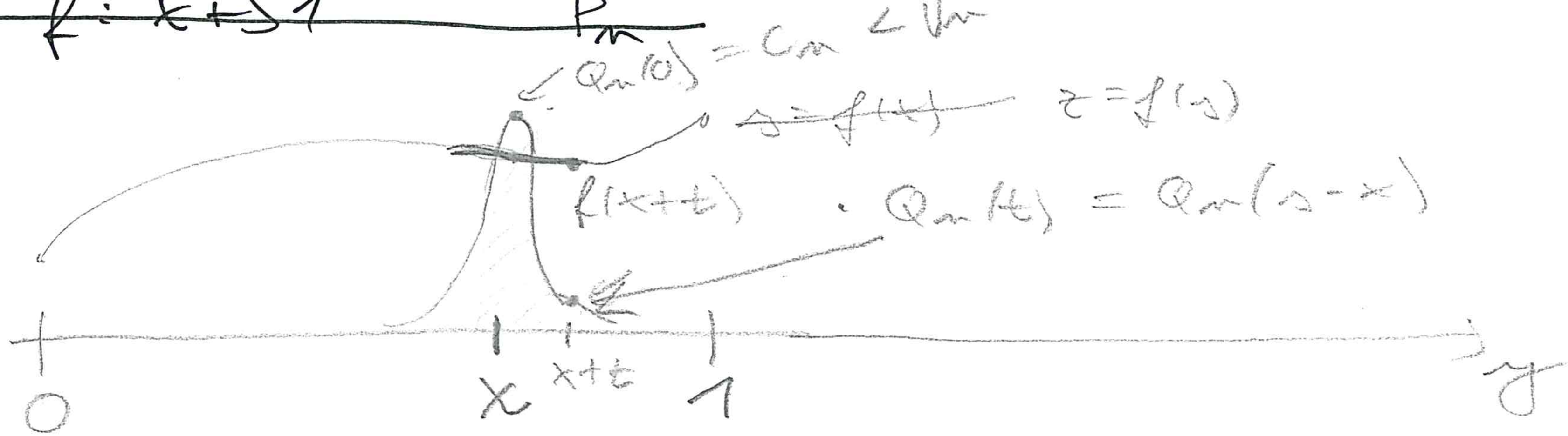
$x+t\in[x-1,x+1] \quad (t\in[-1,1])$

Lépe:

$$P_n(x) = \int_{-x}^{1-x} Q_n(t)\cdot f(x+t)\,dt \qquad (*)$$

Máme zobrazení $\left(f : [0,1] \to \mathbb{R}\right) \mapsto \left(P_n : [\ ?\ ] \to \mathbb{R}\right)$

~~Př: $f : x \mapsto 1$ P_n~~



Cíl je ukázat:

1) pro velká n je $|f(x) - P_n(x)|$ malé $x \in [0,1]$

přesněji $(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall x \in [0,1])$ $\left(|f(x) - P_n(x)| < \varepsilon\right)$

2) P_n je polynom

Substituce v (*) : $y = x + t$ $\quad dy = dt$

$$P_n(x) = \int_0^1 Q_n(y-x)\, f(y)\, dy =$$

$$= \int_0^1 c_n \left(1-(y-x)^2\right)^n f(y)\, dy =$$

$$= c_n \int_0^1 \text{polynom proměnné } x \text{ stupně } 2n \text{ s koeficienty závislými na } y \; f(y)\, dy$$

$n=1$ $\quad 1-(y-x)^2 = -x^2 + 2xy + 1 - y^2$

$n=2$ $\quad \left(1-(y-x)^2\right)^2 = x^4 - 4x^3y + x^2\left(-2(1-y^2) + 4y^2\right) + \dots$

Tedy $P_n(x)$ je polynom

$$f(x) - P_n(x) = f(x) - \int_0^1 Q_n(y-x) f(y)\,dy =$$

$$= \int_0^1 Q_n(y-x) \left(\cancel{f(y) - f}\right) \left(f(x) - f(y)\right) dy \quad ?$$

$$f(x) = f(x) \cdot 1 = f(x) \int_{-1}^{1} Q_n(t)\,dt = \int_{-1}^{1} f(x)\, Q_n(t)\,dt$$

Rozšíříme funkci Q_n:

$$Q_n(t) = \begin{cases} c_n(1-t^2)^n & t \in [-1,1] \\ 0 & t \notin [-1,1] \end{cases}$$

Pak $$\int_{-\infty}^{+\infty} Q_n(t)\,dt = 1 \qquad \left(\int_{-\infty}^{+\infty} Q_n(y-x)\,dy = 1 \right)$$

$t = y - x$

$$f(x) = \int_{-\infty}^{+\infty} f(x)\, Q_n(y-x)\,dy$$

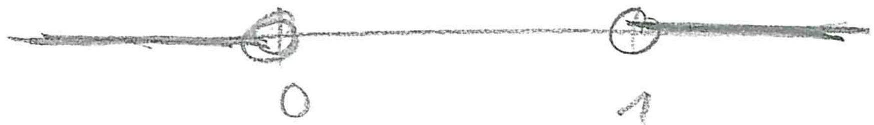
Rozšíříme funkci f:

$$f(y) = \begin{cases} f(y) & y \in [0,1] \\ 0 & y \notin [0,1] \end{cases}$$

Pak

$$f(x) - P_n(x) = \int_{-\infty}^{+\infty} Q_n(y-x)\big(f(x) - f(y)\big)\,dy$$

$$\underbrace{\int_{-\infty}^{+\infty} Q_n(y-x) f(x)\,dy}_{= f(x)} - \underbrace{\int_{-\infty}^{+\infty} Q_n(y-x) f(y)\,dy}_{= \int_0^1 Q_n(y-x) f(y)\,dy}$$



$$|f(x) - P_n(x)| = \left| \int_{-\infty}^{+\infty} Q_n(y-x)\,(f(y) - f(x))\,dy \right| \le$$

$y - x = 1$

$y - x = -1$

$$\le \int_{\cancel{-\infty}\; x-1}^{\cancel{+\infty}\; x+1} |Q_n(y-x)| \cdot |f(y) - f(x)|\,dy$$

Věta:

Je-li f Riemannovsky integrovatelná na $[a,b]$,

pak $$\left| (R)\int_a^b f(x)\,dx \right| \le (R)\int_a^b |f(x)|\,dx$$

Později uděláme důkaz,

zde obrázek:

pro $y-x \sim 0$ je $Q_n(y-x) \sim c_n$ ''

$$|f(y)-f(x)| \sim 0$$

pro ostatní x,y je $Q_n(y-x) \sim 0$

||

$$\int_{x-1}^{x+1} |Q_n(y-x)| \cdot |f(y)-f(x)| \, dy = \int_{x-1}^{x-\delta} \ldots + \int_{x-\delta}^{x+\delta} \ldots + \int_{x+\delta}^{x+1} \ldots$$

$y \in [x-1, x-\delta]$ ... $y-x \in [-1, -\delta]$ ... $|Q_n(y-x)| \le c_n (1-\delta^2)^n$

$y \in [x-\delta, x+\delta]$

$|f(y)-f(x)|$ je malé

ukážeme, že je malé