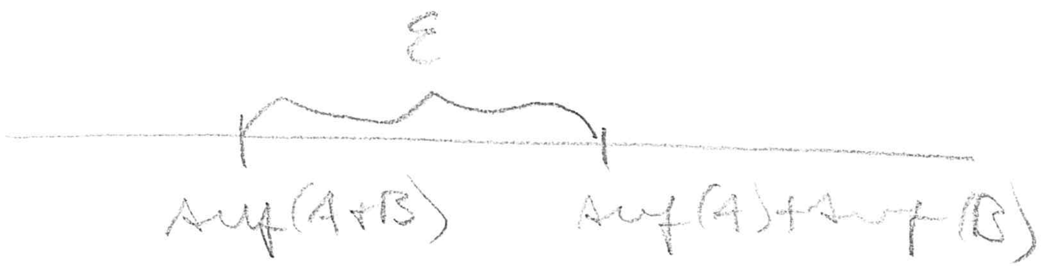
$$A+B = \{a+b \in \mathbb{R} : a \in A,\ b \in B\}$$

$$\sup(A+B) \geq \sup(A) + \sup(B)$$

Sporem: předpokládáme

$$\sup(A+B) < \sup(A) + \sup(B)$$



$$\varepsilon = \sup(A) + \sup(B) - \sup(A+B) > 0$$

$$a \in A : \quad a > \sup(A) - \frac{\varepsilon}{2}$$

$$b \in B : \quad b > \sup(B) - \frac{\varepsilon}{2}$$

$$a + b > \underbrace{\sup(A) + \sup(B) - \varepsilon}_{\sup(A+B)}$$

$$a+b \in A+B \qquad \lightning$$

Při důkazu jsme předpokládali, že A, B jsou shora omezené – $\sup(A), \sup(B) \in \mathbb{R}$ a neprázdné

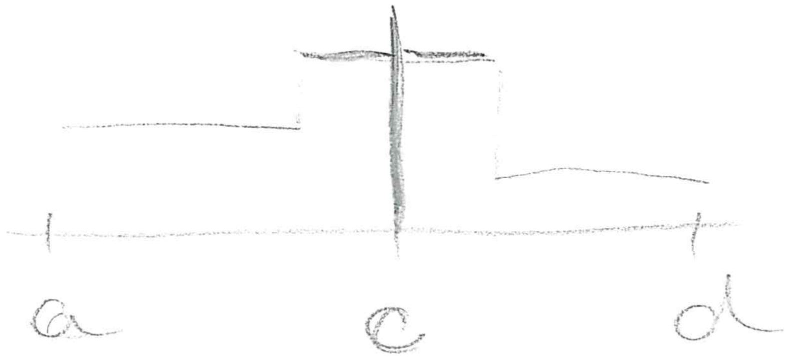
Dokončení důkazu (ŠR)

A ... množina DIS na (a,c)

B ... — " — (c,b)

C — " — (a,b)

$A+B \subseteq C$ $\quad (\forall a\in A)(\forall b\in B)(a+b\in C)$



$C \subseteq A+B$ $\quad (\forall c\in C)(\exists a\in A)(\exists b\in B)(c=a+b$

Závěr: $C = A+B$

tedy $\sup(C) = \sup(A+B) = \sup(A)+\sup(B)$

$$\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$$

Pro rovnost

$$\overline{\int_a^b} f(x) = \overline{\int_a^c} f(x) + \overline{\int_c^b} f(x)$$

---

potřebujeme

$$\inf(A+B) = \inf(A) + \inf(B)$$

důkaz – D.Ú

Předpokládejme: f je R-integrovatelná na $[a,c]$
na $[c,b]$:

$$\underline{\int_a^c} f(x) = \overline{\int_a^c} f(x) \qquad \underline{\int_c^b} f(x) = \overline{\int_c^b} f(x)$$

odtud

$$\underline{\int_a^b} f(x) = \overline{\int_a^b} f(x)$$

tedy f je R-integrovatelná na $[a,b]$

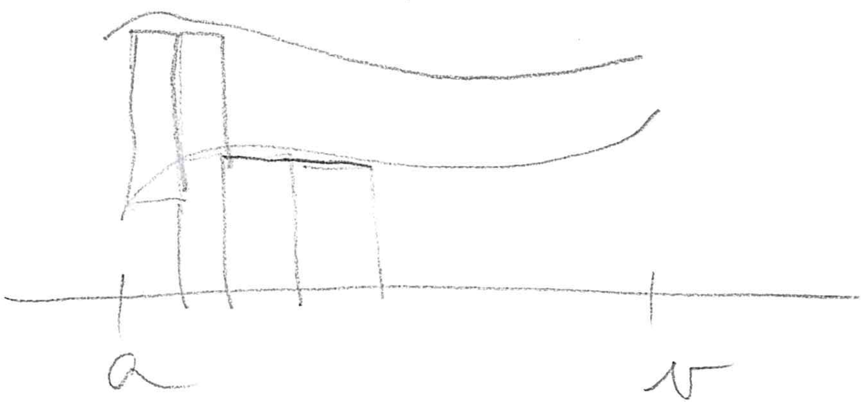
a $$\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$$

Důkaz (2R)

$$\int_a^b f(x)+g(x) = \int_a^b f(x) + \int_a^b g(x)$$

F .. množina DIS f na (a,b)

G .. — " ~~g~~ — " —



~~f ∈ F, g ∈ G .. pak f+g je DIS~~

a ∈ F, b ∈ G, pak a+b je DIS funkce f+g

H .. množina DIS f+g na (a,b)

→ F+G ⊆ H

H ⊆ F+G obecně neplatí

$f(\mathbb{Q})=1$ $g(\mathbb{I})=1$

$f(\mathbb{I})=0$ $g(\mathbb{Q})=0$

Analyzujme: $F+G \subseteq H$

$$\sup(F) + \sup(G) \ \overset{?}{} \ \sup(H)$$

Tvrdíme: $\underbrace{\sup(F) + \sup(G)}_{=\sup(F+G)} \leq \sup(H)$ (*)

$$A \subseteq B \Rightarrow \sup(A) \leq \sup(B)$$

($\sup(B)$ je h. závor. A)

HLS: F, G, H .. množiny HLS

~~inf~~

$$F+G \subseteq H$$

$$\inf(F+G) = \inf(F) + \inf(G)$$

$$\inf(F+G) \geq \inf(H) \quad (**)$$

předpokládáme:

$$\underline{\int_a^b} f(x) = \overline{\int_a^b} f(x)$$

$$\underline{\int_a^b} g(x) = \overline{\int_a^b} g(x)$$

$$\underline{\int_a^b} f(t) + \underline{\int_a^b} g(t) \leq \underline{\int_a^b} f(t)+g(t) \qquad z\,(*)$$

$$\leq$$

$$= \overline{\int_a^b} f(t) + \overline{\int_a^b} g(t) \geq \overline{\int_a^b} f(t)+g(t) \qquad z\,(**)$$

$$\leq$$

odtud
$$\underline{\int_a^b} f(t)+g(t) = \overline{\int_a^b} f(t)+g(t)$$

tedy $f+g$ je R-integrovatelná na $\langle a, b\rangle$

a platí

$$\int_a^b f(t)+g(t) = \int_a^b f(t) + \int_a^b g(t)$$

dintar (2N)

$$\int_a^b f(x) = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x) \in \mathbb{R}$$

$$\int_a^b g(x) = \lim_{x\to b^-} G(x) - \lim_{x\to a^+} G(x) \in \mathbb{R}$$

$$(F+G)' = f+g$$

[illegible]

$$\int_a^b f(x)+g(x) = \lim_{x\to b^-} (F(x)+G(x)) - \lim_{x\to a^+} (F(x)+G(x))$$

$$= \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x) + \lim_{x\to b^-} G(x) - \lim_{x\to a^+} G(x) =$$

$$= \int_a^b f(x) + \int_a^b g(x)$$

4N

$(\forall x \in (a,b))(f(x) \geq 0)$, potom je

F neklesající, pak

$$\lim_{x \to b^-} F(x) = \sup \{ F(x) : x \in (a,b) \}$$

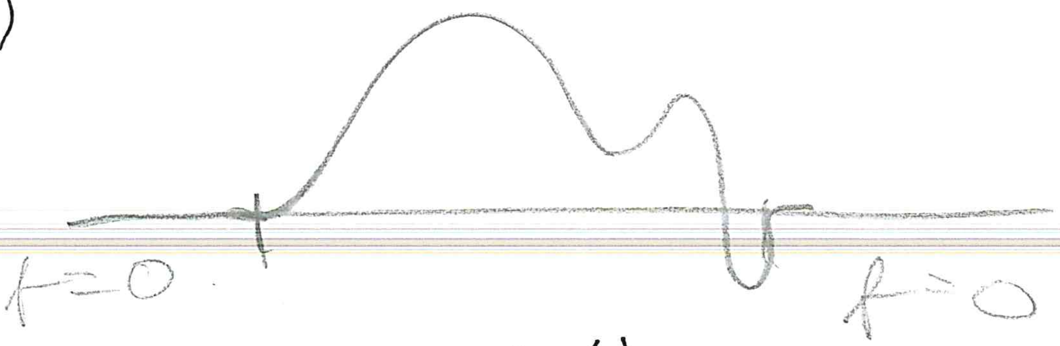
$$\lim_{x \to a^+} F(x) = \inf \{ F(x) : x \in (a,b) \}$$

odtud: $\lim_{x \to a^+} F(x) \leq \lim_{x \to b^-} F(x)$

a tedy $\int_a^b f(x) \geq 0$

D.Ú 4R, 3N, 3R (1R, 1N)

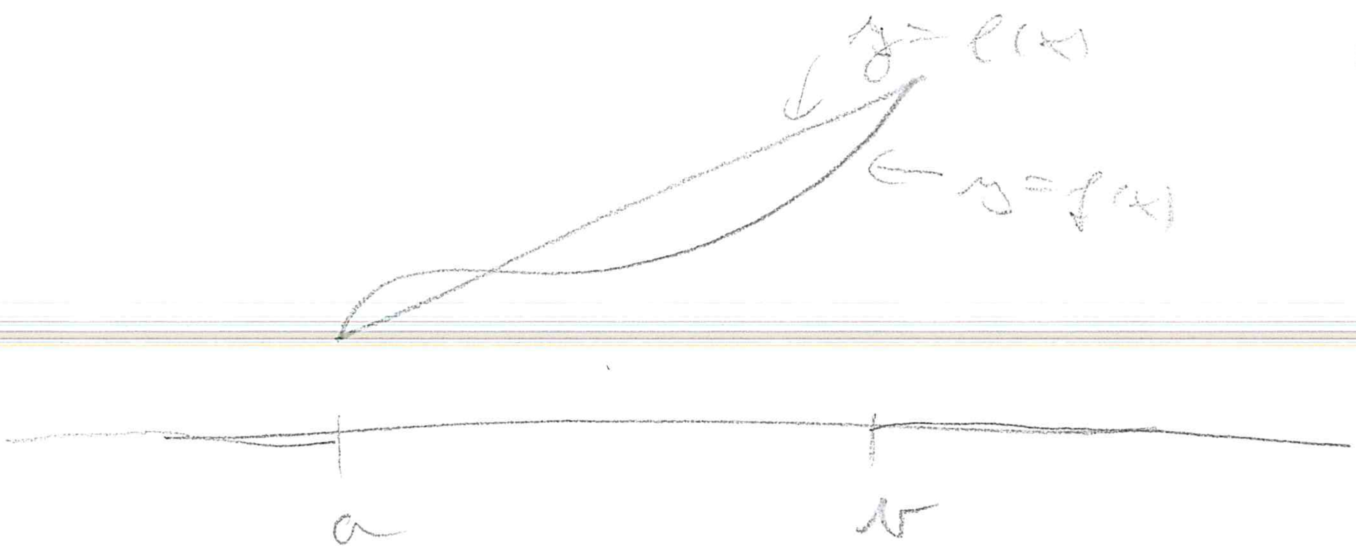
$C_c^\infty(\mathbb{R})$



$x \mapsto \exp\left(-\frac{1}{x^2}\right)$ $\times$

$$x \mapsto \begin{cases} \exp\left(\frac{-1}{1-x^2}\right) & x \in (-1,1) \\ 0 & x \notin (-1,1) \end{cases}$$

9



$g(x) = f(x) - \ell(x)$

$g(a) = g(b) = 0$

6N

6R

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

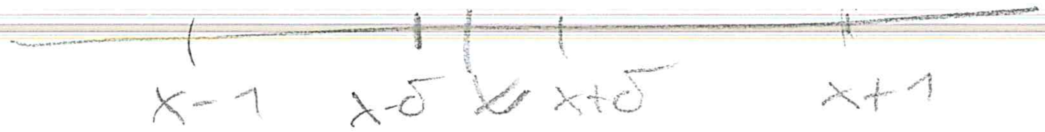
wiemy: $-|f(x)| \le f(x) \le |f(x)|$ dla $x \in (a,b)$

stąd

$$-\int_a^b |f(x)| \le \int_a^b f(x) \le \int_a^b |f(x)|$$

stąd $\left|\int f(x)\right| \le \int |f(x)|$

$(-|a| \le b \le |a| \Leftrightarrow |b| \le |a|)$



$$\int_{x-1}^{x-\delta} |Q_n(y-x)|\cdot|f(x)-f(y)|\,dy \; + \quad = I_1$$

$$+ \int_{x-\delta}^{x+\delta} \text{—}\,''\,\text{—} \; + \quad = I_2$$

$$+ \int_{x+\delta}^{x+1} \text{—}\,''\,\text{—} \quad = I_3$$

chceme: $(\forall \varepsilon > 0)(\exists n)(\exists \delta > 0)\left(I_1 < \frac{\varepsilon}{3},\ I_2 < \frac{\varepsilon}{3},\ I_3 < \frac{\varepsilon}{3}\right)$

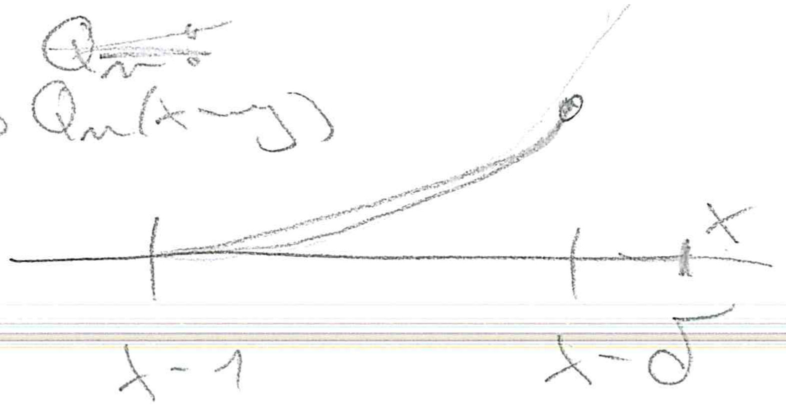
víme: $|Q_n(y-x)| < \sqrt{n}$

$|f(x)-f(y)| \leq M$

(f je omezená na $[a,b]$, $|f(x)| \leq M$,

$|f(x)-f(y)| = |f(x)+(-f(y))| \leq |f(x)|+|-f(y)| \leq 2M$)

$y \mapsto Q_n(x-y)$



pro $y \in [x-1, x-\delta]$ je

$$|Q_n(x-y)| \le Q_n(x-(x-\delta)) = Q_n(\delta) = c_n(1-\delta^2)^n$$

zvolme n, aby

$$c_n(1-\delta^2)^n \cdot M \cdot (1-\delta) < \frac{\varepsilon}{3}$$

1) je možné n takto zvolit pro každé $\varepsilon > 0$?

ano, protože $\underbrace{c_n(1-\delta^2)^n \xrightarrow[n\to\infty]{} 0}_{\text{D.Ú}}$

2) tak $$\int_{x-1}^{x-\delta} |Q_n(x-y)|\,|f(x)-f(y)|\,dy < \frac{\varepsilon}{3}$$

$g \in \mathcal{C}([a,b])$ (spojitá na $[a,b]$)

$h(x) = g(a + (b-a)x)$ je spojitá na $[0,1]$

$$f(x) = \begin{cases} h(x) - (x\,h(1) + (1-x)\,h(0)) & \text{pro } x \in [0,1] \\ 0 & \text{jinak} \end{cases}$$

je spojitá na $\mathbb{R}$ $(f \in \mathcal{C}(\mathbb{R}))$

f je stejnoměrně spojitá na každém uzavřeném (omezeném) intervalu, zvolíme $[-1,2]$, pak

$$(\forall \varepsilon > 0)(\exists \delta \in (0,1))(\forall x,y \in [-1,2])\left(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{4}\right)$$

Pak je pro $x \in [0,1]$:

$$\int_{x-\delta}^{x+\delta} |Q_n(x-y)|\,|f(x) - f(y)|\,dy \le \int_{x-\delta}^{x+\delta} |Q_n(x-y)| \cdot \frac{\varepsilon}{4}\,dy \le$$

$$\le \frac{\varepsilon}{4} \int_{\mathbb{R}} Q_n(x-y)\,dy = \frac{\varepsilon}{4} < \frac{\varepsilon}{3}$$

k $\varepsilon>0$, $\delta>0$ pak volíme n takové, že

$$\sqrt{n}\,(1-\delta^2)^n H(1-\delta) < \frac{\varepsilon}{3}$$

existence takového n plyne z

$$\lim_{n\to\infty} \sqrt{n}\,(1-\delta^2)^n = 0 \quad \text{pro } \delta\in(0,1).$$

Pro tato n, δ, ε je

$$\int_{\mathbb{R}} |Q_n(x-y)|\,|f(x)-f(y)|\,dy = I_1+I_2+I_3 < \varepsilon$$

Poznámka: podstatné je, že n závisí na ε a nikoliv na X.

Poznámka: dokázali jsme, že pro $f \geq 0$ na (a,b) platí $\int_a^b f \geq 0$, ale

po úpravě textu už ne

potřebujeme pro $f > 0$ na (a,b) ostrost $\int_a^b f > 0$.

To pro spojitou funkci plyne z: $f(x_0) > 0$ .. $\exists \delta > 0$, že pro $x \in U_\delta(x_0)$ je $f(x) > \frac{f(x_0)}{2}$

a odtud

$$\int_R f(x)\,dx \geq \int_{U_\delta(x_0)} f(x)\,dx \geq \delta f(x_0) > 0$$