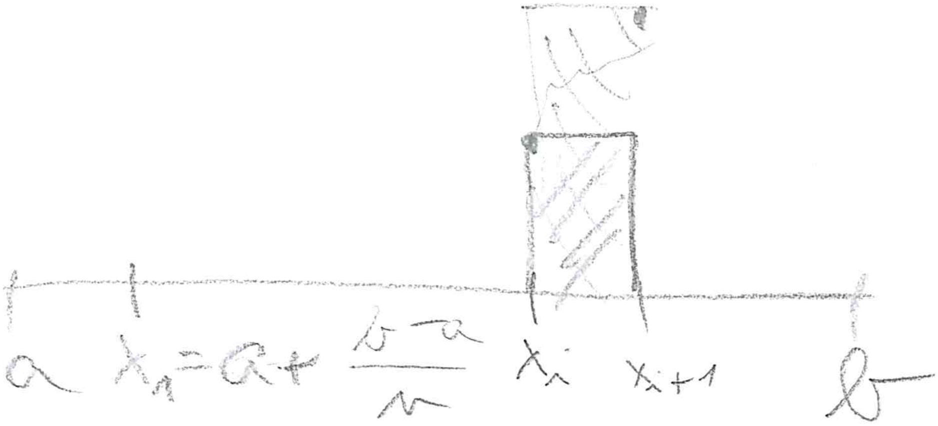
Věta:

Nechť $f$ je neklesající na $[a,b]$, pak je $f$ Riemannovsky integrovatelná na $[a,b]$.

Důkaz:




$$x_i = a + i\,\frac{b-a}{n}\,, \quad i = 0, \dots, n$$

$$DIS = \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i)$$

$$HIS = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

$$HIS - DIS = \frac{b-a}{n}\Big(f(x_n) - f(x_0)\Big) =$$

$$= \frac{b-a}{n}\Big(f(b) - f(a)\Big) < \varepsilon$$

1) $f(b) - f(a) = 0$ funkce je konstant-

2) $f(b) - f(a) > 0$, $b - a > 0$

$$n > \underbrace{(b-a)}_{>0}\underbrace{(f(b) - f(a))}_{>0} / \underbrace{\varepsilon}_{>0}$$

$$n = \lceil (b-a)(f(b) - f(a)) / \varepsilon \rceil + 1$$

~~Na~~ Existuje DIS, HIS splňující

$$HIS - DIS < \varepsilon$$

Podmínka lemma o riemannovské integrovatelnosti.

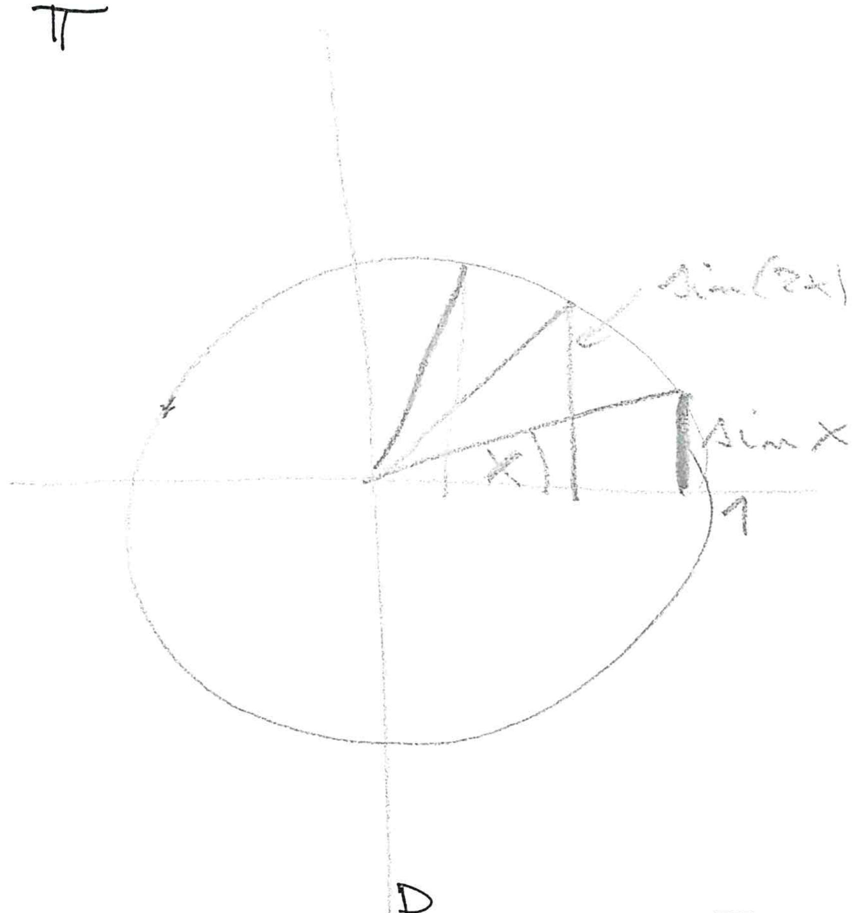
$\square$

$$\sum_{n=1}^{\infty} \sin(nx)$$

1) $\cancel{x \notin \mathbb{Q},\ x \neq \frac{p}{q}}$

$$\frac{x}{\pi} \in \mathbb{Q}, \quad x = \frac{p}{q} \cdot \pi$$

$$nx = \frac{pn}{q} \pi$$




$$n = 2q: \quad nx = 2q \cdot \frac{p}{q} \cdot \pi = 2p\pi$$

$$\sin(nx) = \sin(2p\pi) = 0$$

$$\sin((n+1)x) = \sin(x)$$

$\sin x = 0$ .. $x = k\pi$, $k \in \mathbb{Z}$

$\cancel{x = 2\pi}$

②

1a) $x = k\pi$, $k \in \mathbb{Z}$

$$\sum_{n=1}^{\infty} \sin(nx) = \sum_{n=1}^{\infty} 0 = 0$$

1b) $\frac{x}{\pi} \in \mathbb{Q} / \mathbb{Z}$

$$\sin((2g+1)x) = \sin(x) \neq 0$$

$\varepsilon = \frac{|\sin(x)|}{2}$, pro nekonečně

mnoho indexů $n$ je $|\sin(nx)| > \varepsilon$

není splněna nutná podmínka

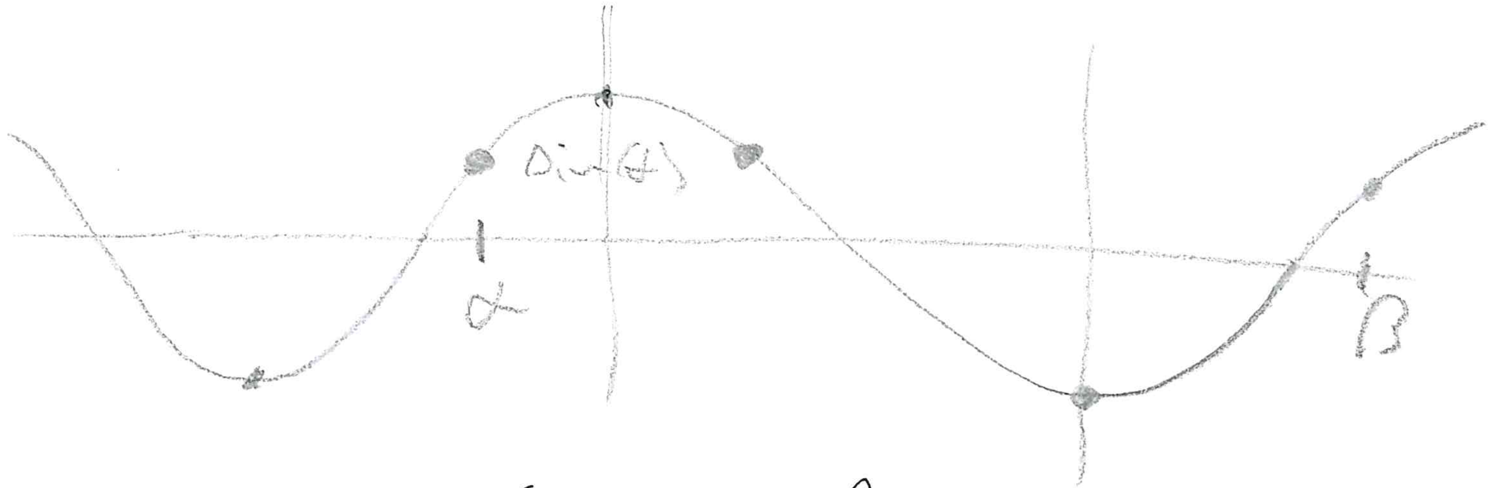
konvergence

závěr: řada nekonverguje

2) $\frac{x}{\pi} \notin \mathbb{Q}$

může být $\sin(nx) = \sin(mx)$

pro $n \neq m$ ?




$\sin(\alpha) = \sin(\beta)$ — $\frac{\beta - \alpha}{2\pi} \in \mathbb{Z}$

$$\frac{\alpha+\beta}{2} = \frac{\pi}{2} + k\pi$$

$$\frac{\alpha + (\beta - 2\ell\pi)}{2} = \frac{\pi}{2} + k\pi \quad / \cdot 2$$

$$\alpha + \beta - 2\ell\pi = \pi + 2k\pi$$

$$\boxed{\begin{aligned} \alpha + \beta &= \pi(1 + 2k + 2\ell) \\ \beta - \alpha &= 2k\pi \end{aligned}}$$

$\alpha = nx$, $\beta = mx$

$\alpha + \beta = (n+m)x = \pi(1 + 2k + 2l)$

$$x = \pi \frac{1+2k+2l}{n+m} \in \mathbb{Q}$$

$$\frac{x}{\pi} \in \mathbb{Q}$$

$\beta - \alpha = mx - nx = (m-n)x$

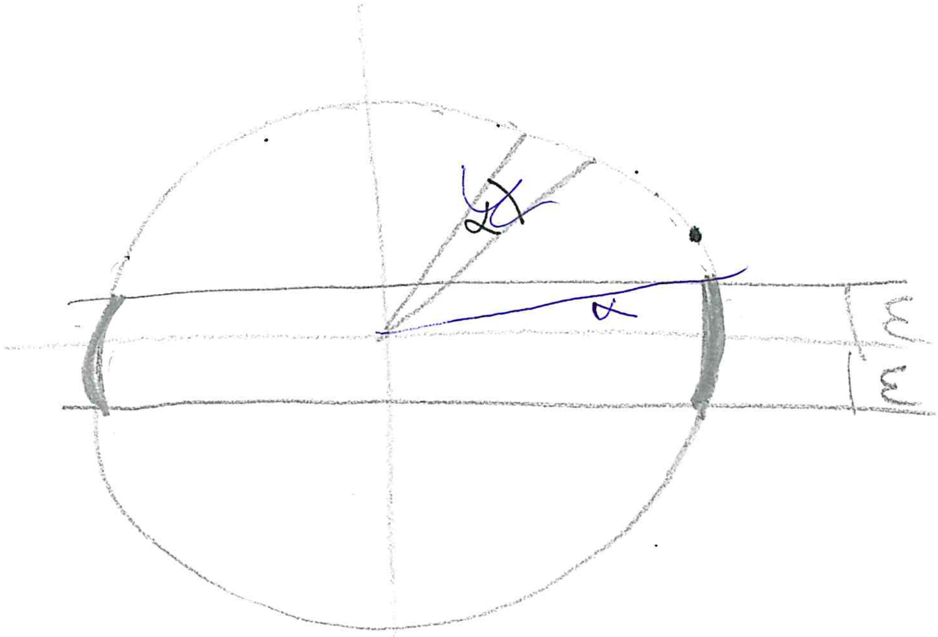
předpokládáme: $m \neq n$

$$x = \frac{2k\pi}{m-n} \mathbb{Q}$$

$$\frac{x}{\pi} \in \mathbb{Q}$$

Ukážeme, že pro $\frac{x}{\pi} \notin \mathbb{Q}$ není splněna nutná podmínka konvergence.

Sporem: zvolme $\varepsilon > 0$




Ukážeme: ~~...~~ $\cancel{(\exists\, m, n \in \mathbb{N},\ m \neq n)\ (|mx - nx| < \frac{\pi}{2})}$

$\cancel{(\exists\, m \in \mathbb{N},\ m \neq 0)\ (|mx| < \frac{\pi}{2})}$

$\cancel{|mx-}$

$$\left\lfloor \frac{mx}{2\pi} \right\rfloor = k \qquad \alpha|\underbrace{mx - 2k\pi}_{\in [0, 2\pi)}| < \alpha\}$$

$$mx = 2k\pi + \beta, \quad k \in \mathbb{Z} \qquad \frac{mx}{2\pi} = k + \frac{\beta}{2\pi} \in [0,1)$$

vor: $0 < |nx - 2k\pi| < \alpha$ pro zvolené malé $\alpha$
$k \in \mathbb{Z}$

~~$|\sin(\alpha)| < \varepsilon$~~

~~$\sin\beta \in (-\varepsilon, \varepsilon)$~~, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

pro $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ je ekvivalentní

$$\sin\beta \in (-\varepsilon, \varepsilon) \iff |\beta| < |nx|$$

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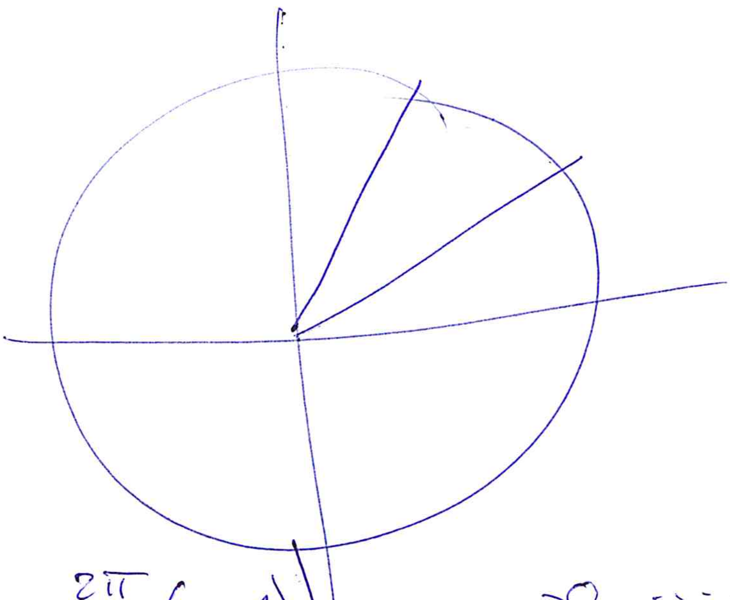
$N = 12$



$$I_n = \left[\frac{2\pi}{12} \cdot n, \frac{2\pi}{12}(n+1)\right), \quad n = 0, \dots, 12$$

$$\alpha_n = [nx]_{2\pi}$$

$$[a]_{2\pi} = a - 2\pi \left\lfloor \frac{a}{2\pi} \right\rfloor \in [0, 2\pi)$$

Uvažuje vybranou posloupnost

$\sin(nx), \sin(2nx) \ldots$

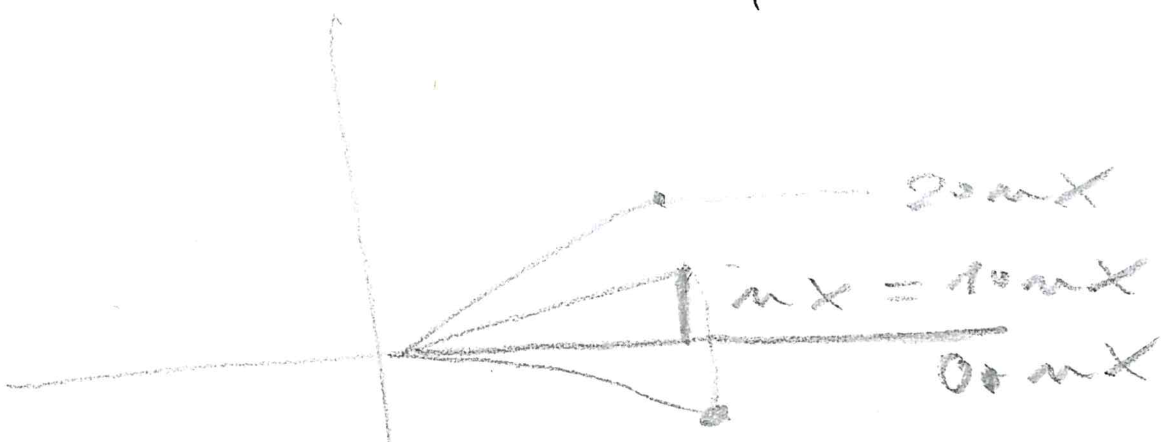
$$\{\sin(nkx)\}_{k=1}^{\infty}$$

z posloupnosti $\{\sin(kx)\}_{k=1}^{\infty}$

z $\lim_{k\to\infty} \sin(kx) = 0$

plyne $\lim_{k\to\infty} \sin(knx) = 0$

volme $\varepsilon = |\sin(nx)|$



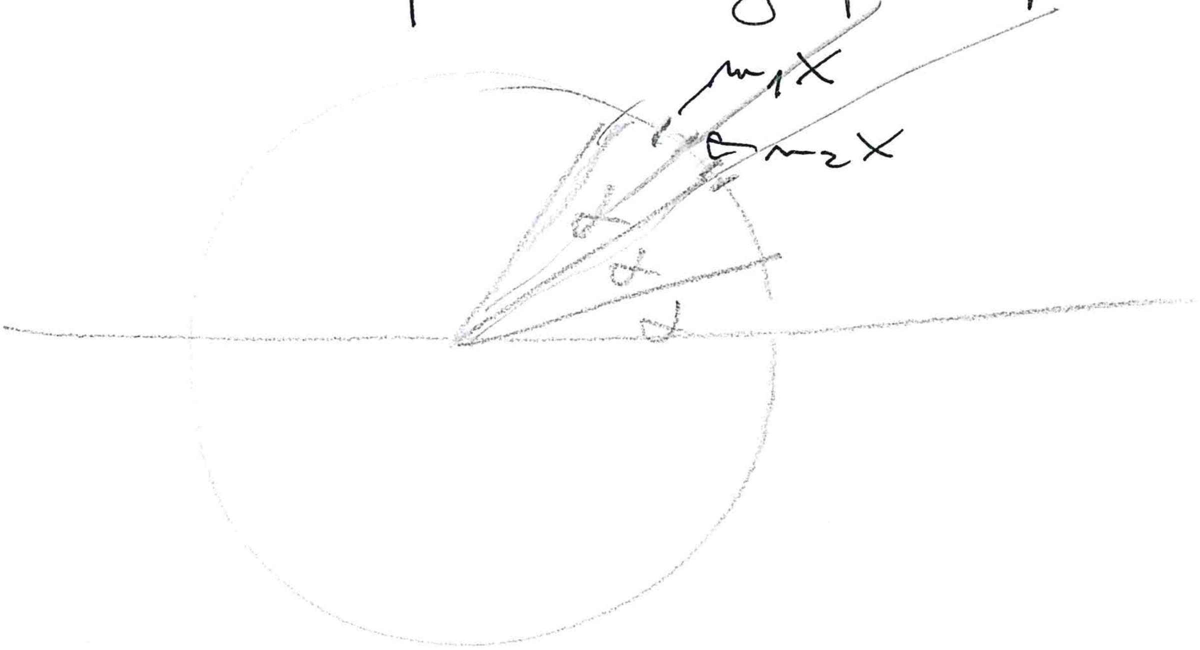

Pak $\sin(nkx) \in (-\varepsilon, \varepsilon)$, tak

$$\sin(\underbrace{nkx + nx}_{nx(k+1)}) \notin (-\varepsilon, \varepsilon)$$

$\frac{x}{\pi} \notin \mathbb{Q}$, $\alpha > 0$

$$(\forall \alpha > 0)(\exists n \in \mathbb{N})\left(0 < \left|nx - 2\pi\left\lfloor\frac{nx}{2\pi}\right\rfloor\right| < \alpha\right)$$

Dirichletův přihrádkový princip




jsou-li $n_1 x$, $n_2 x$ „jsou ve stejné výseči", ~~znač:~~ pak

$$\cancel{\frac{n_1 x - n_2 x}{2\pi}}$$

$$(\exists k \in \mathbb{Z})\left(|n_1 x - n_2 x - 2k\pi| < \alpha\right)$$

$\cancel{|n_1 - n_2|} \equiv n$ zvolme $n_1 > n_2$

$$\cancel{|n|x| - 2k\pi| < \alpha}$$

$$\cancel{|nx + 2k\pi| < \alpha}$$

odtud: $n = n_1 - n_2$

$$0 < |nx - 2k\pi| < \alpha$$