Definice:

Nechť $a, b \in \mathbb{R}$.

Řekneme, že a je kongruentní s b modulo 2π, pokud $\frac{a-b}{2\pi} \in \mathbb{Z}$.

Budeme značit $a \equiv b$.

Lemma:

Nechť $a \equiv b$, $c \equiv d$, $n \in \mathbb{N}$.

Pak $a+c \equiv b+d$,

$a-c \equiv b-d$,

$na \equiv nb$

$\sin(a) = \sin(b)$

Důkaz:

$$\frac{a+c-(b+d)}{2\pi} = \frac{a-b}{2\pi} + \frac{c-d}{2\pi} \in \mathbb{Z}$$

$$\frac{a-c-(b-d)}{2\pi} = \frac{a-b}{2\pi} - \frac{c-d}{2\pi} \in \mathbb{Z}$$

$$\frac{na-nb}{2\pi} = n\frac{a-b}{2\pi} \in \mathbb{Z}$$

$$\sin(a) = \sin\left(b + 2\pi\frac{a-b}{2\pi}\right) = \sin(b)$$

$\square$

Definice:

Nechť $a \in \mathbb{R}$, $I \subseteq \mathbb{R}$.

$a \in I$ bude značit $(\exists b \in I)(a \equiv b)$

Lema:

$$(\forall a \in \mathbb{R})\Big(a \in [0, 2\pi) \\ a \in (-\pi, \pi] \\ a \in \left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right] \Big)$$

Definice:

Řekneme, že posloupnost $\{x_n\}$, $x_n \in \mathbb{R}$ je $\equiv$prostá, pokud z $x_n \equiv x_m$ plyne $n=m$.

Lemma:

Nechť $x \in \mathbb{R}$, $\frac{x}{\pi} \notin \mathbb{Q}$.

Pak je posloupnost $\{nx\}_{n=0}^{\infty}$ $\equiv$prostá.

Důkaz:

$nx \equiv mx$ znamená:

$$\frac{nx - mx}{2\pi} = \frac{x}{\pi} \cdot \frac{n-m}{2} \in \mathbb{Z}$$

odkud a z $\frac{x}{\pi} \notin \mathbb{Q}$ plyne $n = m$.

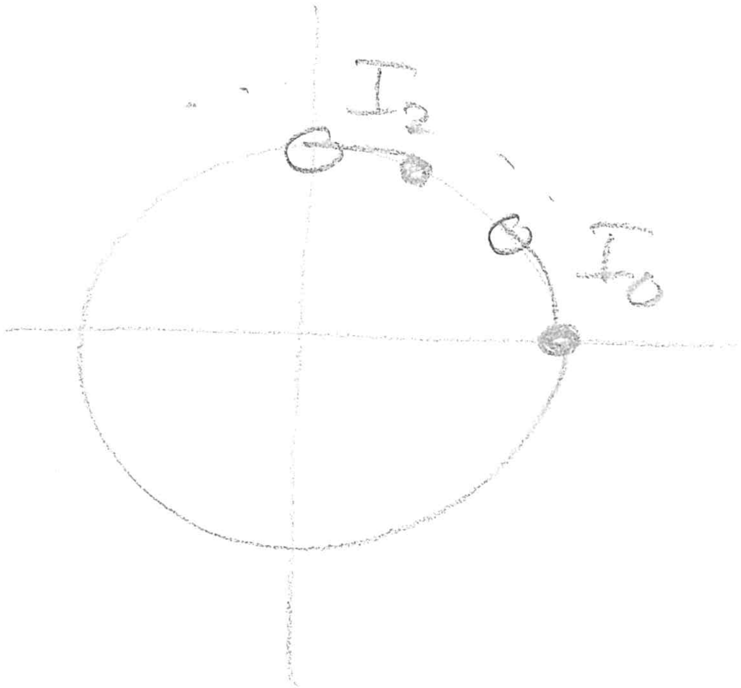
$\square$

Dirichletův princip

$$I_n = \left[n\frac{\pi}{6}, (n+1)\frac{\pi}{6}\right), \quad n \in \{0, 1, \dots, 11\}$$

Existují $k, l \in \{0, \dots, 12\}$, $n \in \{0, \dots, 11\}$, $k \neq l$

takové, že $kx \in I_n$, $lx \in I_n$



$kx \equiv a \in I_n$ BÚNO $k > L$

$lx \equiv b \in I_n$

$kx - lx \equiv a - b$

pro $\frac{x}{\pi} \notin \mathbb{Q}$ je $mx \not\equiv 0$

$m = k - l \in \mathbb{N}$

$mx = a - b \in \left(-\frac{\pi}{6}, 0\right) \cup \left(0, \frac{\pi}{6}\right)$

BÚNO $mx \in \left(0, \frac{\pi}{6}\right)$

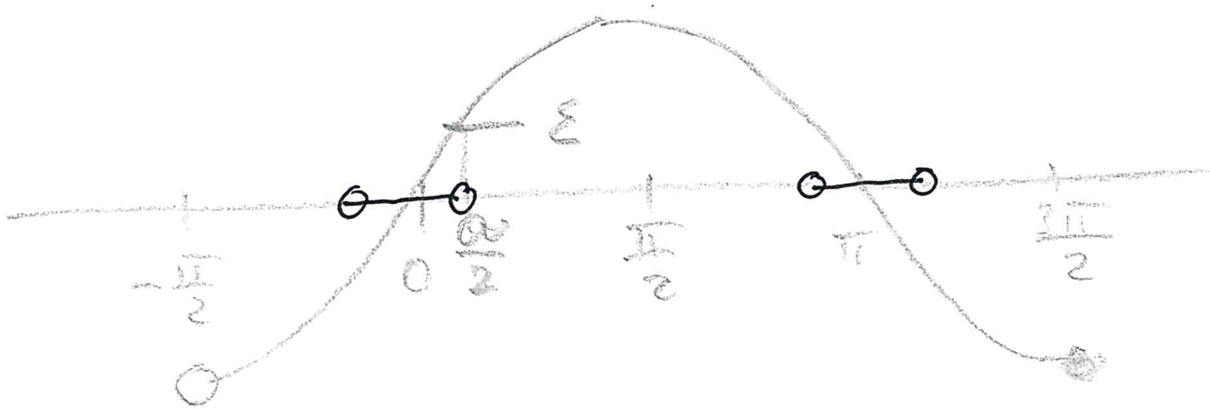
[jinak $y = -x$ $\cancel{\in \left(0, \frac{\pi}{6}\right)}$, $my \in \left(0, \frac{\pi}{6}\right)$

$\lim_{m \to +\infty} (mx) = 0 \Leftrightarrow \lim_{m \to +\infty} (my) = 0$]

$nx \in (0, \frac{\pi}{6})$

$nx \equiv a \in (0, \frac{\pi}{6})$

$\varepsilon = \sin\frac{a}{2} > 0$



$|\sin(mx)| < \varepsilon \iff mx \in (-\frac{a}{2}, \frac{a}{2}) \cup (\pi - \frac{a}{2}, \pi + \frac{a}{2})$

pak $(n+m)x = nx + mx \in (\frac{a}{2}, \frac{3a}{2}) \cup (\pi + \frac{a}{2}, \pi + \frac{3a}{2})$

a tedy
$$|\sin(nx + mx)| > \varepsilon$$

odtud plyne, že není splněna nutná podmínka konvergence pro řadu $\sum_{n=1}^{\infty} \sin(nx)$

$$mx \in \underbrace{\left(-\frac{a}{2}, \frac{a}{2}\right) \cup \left(\pi - \frac{a}{2}, \pi + \frac{a}{2}\right)}_{M}$$

$$(\exists b \in M)(b \equiv mx)$$

$$nx \equiv a$$

$$mx \equiv b$$

$$(n+m)x \equiv a+b \in \left(\frac{a}{2}, \frac{3a}{2}\right) \cup \left(\pi + \frac{a}{2}, \pi + \frac{3a}{2}\right)$$

$$\cancel{a \in \left(0, \frac{\pi}{6}\right)}$$

$$b \in \left(-\frac{a}{2}, \frac{a}{2}\right) \cup \left(\pi - \frac{a}{2}, \pi + \frac{a}{2}\right)$$

pak $a+b \in \left(\frac{a}{2}, \frac{3a}{2}\right) \cup \left(\pi + \frac{a}{2}, \pi + \frac{3a}{2}\right)$

$$x \in (A, B)$$

$$x + C \in (A+C, B+C)$$