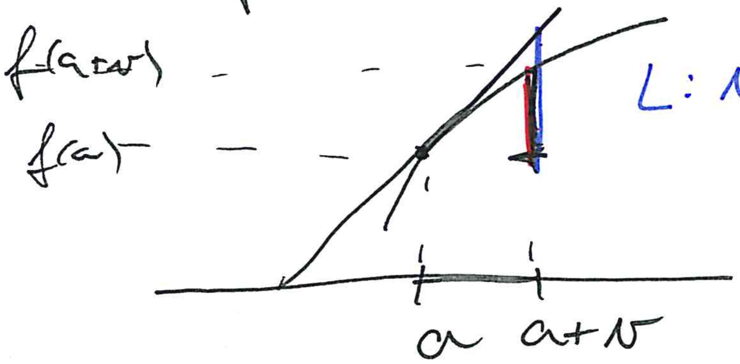
Definice:

Derivací (silnou derivací) funkce $f: \mathbb{R}^2 \to \mathbb{R}^1$ v bodě $a \in \mathbb{R}^2$ nazýváme lineární zobrazení $L: \mathbb{R}^2 \to \mathbb{R}$ splňující

$$\lim_{v=(v_1,v_2)\to(0,0)} \frac{f(a+v)-f(a)-L(v)}{\|v\|} = 0 \qquad (*)$$

$$\|v\| = \sqrt{v_1^2+v_2^2}$$

Pro $f: \mathbb{R}^1 \to \mathbb{R}^1$:



$L: v \mapsto f'(a)\cdot v$

Věta:

Má-li funkce $f : \mathbb{R}^2 \to \mathbb{R}^1$ v bodě a derivaci, pak je f v bodě a spojitá.

Důkaz:

Z $(*)$ plyne, že limita čitatele je také nula:

$$\lim_{v \to (0,0)} \left(f(a+v) - f(a) - L(v) \right) = 0 \qquad \Big(\text{stačí } \underbrace{\frac{\cdots}{\|v\|}}_{\to 0} \cdot \underbrace{\|v\|}_{\to 0}\Big)$$

$$L(v) = L_1 v_1 + L_2 v_2 \to 0 \quad \text{pro} \quad (v_1, v_2) \to (0,0)$$

odtud

$$\lim_{v \to (0,0)} \left(f(a+v) - f(a) \right) = 0$$

odtud $\lim\limits_{\substack{v \to (0,0) \\ x \to a}} f(\underbrace{a+v}_{x}) = f(a)$

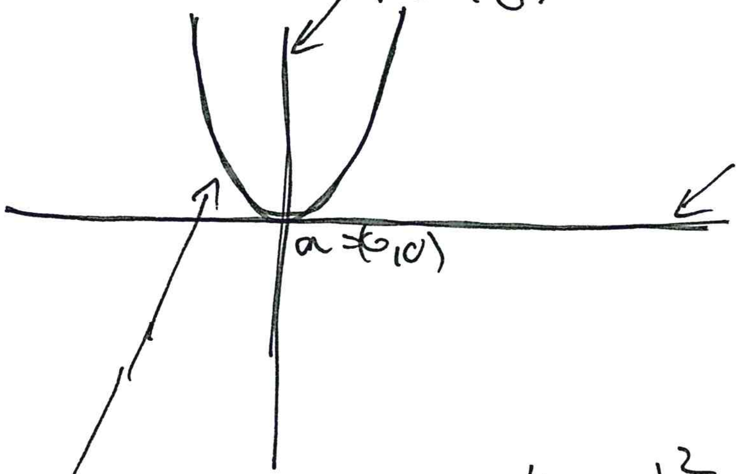
odtud plyne spojitost f v bodě a.

Příklad funkce, která není spojitá v bodě a a má v něm slabou derivaci:

$$f(x,y) = \frac{x^4 y^2}{x^8 + y^4}, \quad a = (0,0)$$

Def. $f(0,0) = 0$

$f(0,y) = 0$



$f(x,0) = 0$

$$f(x,x^2) = \frac{x^4 (x^2)^2}{x^8 + (x^2)^8} = \frac{1}{2}$$

funkce f **není** spojitá v bodě a

$v = (1,0)$ $\quad D_v f(a) = 0$

$v = (0,1)$ $\quad D_v f(a) = 0$

$v = (v_1, v_2)$ $\quad f(a+tv) = f(tv) = \cancel{\dfrac{v_1^4 v_2^2}{v_1^8 + v_2^4}} \quad \dfrac{v_1^4 \cancel{t^4} v_2^2 t^2}{v_1^8 \cancel{t^8}^{t^4} + v_2^4 \cancel{t^4}} =$

$v_1 \neq 0$

$v_2 \neq 0$

$$= \frac{v_1^4 v_2^2 t^2}{v_1^8 t^4 + v_2^4}$$

$$\frac{f(a+tv) - f(a)}{t} = \frac{v_1^4 v_2^2 \cancel{t}}{v_1^8 t^4 + v_2^4} \xrightarrow[t \to 0]{} 0$$

Závěr: $\quad D_v f(a) = 0$ pro $v \in R^2$

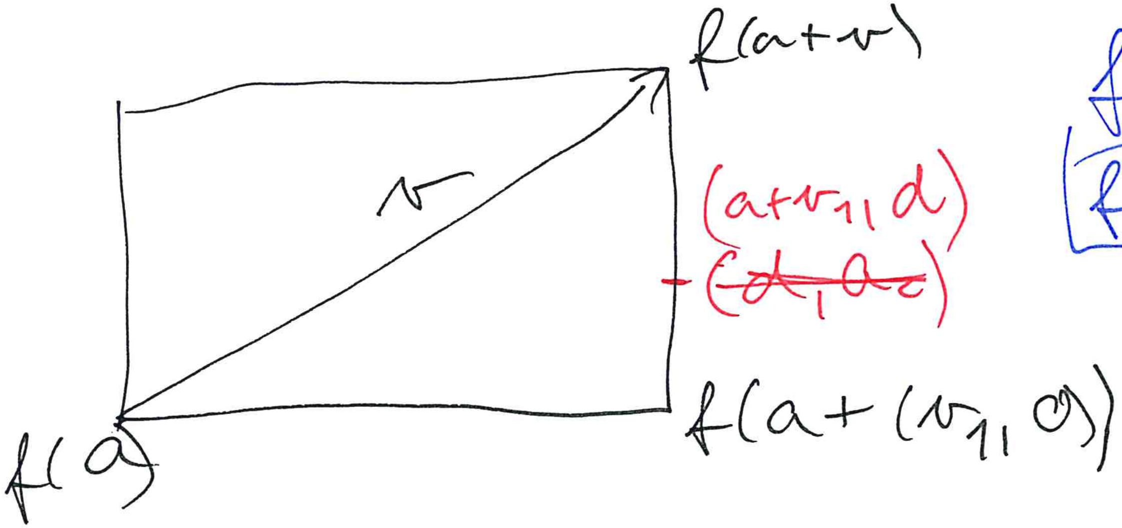
slabá derivace $v \mapsto 0$ (je lineární zobrazení)

f má v bodě a slabou derivaci

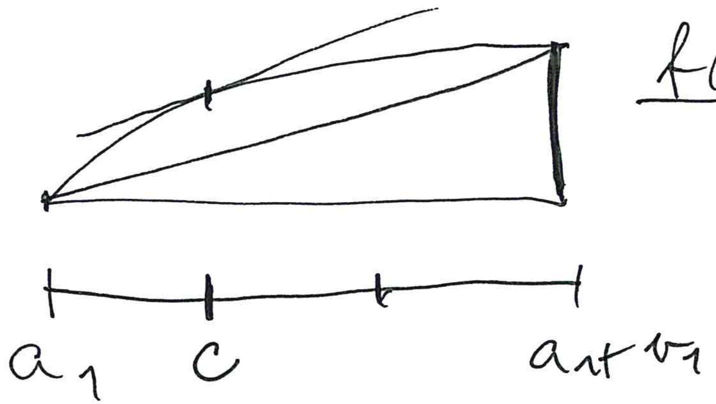
Důkaz věty o existenci silní derivace

$$v \cdot \operatorname{grad} f(a) = \underbrace{v_1 \frac{\partial f}{\partial x}(a) + v_2 \frac{\partial f}{\partial y}(a)}$$

cíl:
$$\frac{f(a+v) - f(a) - L(v)}{\|v\|} \longrightarrow 0 \quad \text{pro} \quad v \to 0$$



$$f(a+v) - f(a) =$$
$$\boxed{f(a+v) - f(a+(v_1,0))} +$$
$$+ \boxed{f(a+(v_1,0)) - f(a)} = \frac{\partial f}{\partial y}(a+v_1, d) \cdot v_2$$



$$\frac{f(a+(v_1,0)) - f(a)}{v_1} = \frac{\partial f}{\partial x}(c, a_2)$$

$$\boxed{f(a+(v_1,0)) - f(a) = v_1 \frac{\partial f}{\partial x}(c, a_2)}$$

$$f(a+v) - f(a) = v_1 \frac{\partial f}{\partial x}(c_1, a_2) + v_2 \frac{\partial f}{\partial y}(a+v_1, \alpha)$$

$$\frac{f(a+v) - f(a) - L(v)}{\|v\|} = \frac{v_1 \frac{\partial f}{\partial x}(c_1, a_2) + v_2 \frac{\partial f}{\partial y}(a+v_1, \alpha) - v_1 \frac{\partial f}{\partial x}(a) - v_2 \frac{\partial f}{\partial y}(a)}{\|v\|} =$$

$$= \frac{v_1}{\|v\|}\left(\frac{\partial f}{\partial x}(c_1, a_2) - \frac{\partial f}{\partial x}(a)\right) + \frac{v_2}{\|v\|}\left(\frac{\partial f}{\partial y}(a+v_1, \alpha) - \frac{\partial f}{\partial y}(a)\right)$$

$\to 0$ pro $v \to o$

omezené

$\to 0$ pro $v \to o$

$$\frac{v_1}{\|v\|} \in [-1, 1]$$

$$-1 \leq \frac{v_2}{\|v\|} \leq 1$$

Součin omezené funkce a funkce s nulovou limitou má limitu rovnu nule.