Dvojný integrál $$\iint_M f(x,y)\,dx\,dy$$

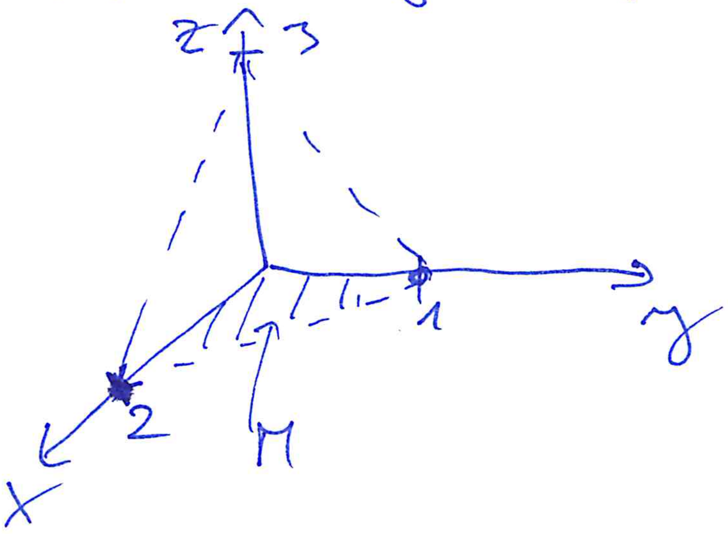
Dvojnásobný integrál $$\int_{\mathbb{R}} \left(\int_{M_{*,y}} f(x,y)\,dx \right) dy$$

$$\int_{\mathbb{R}} \left(\int_{M_{x,*}} f(x,y)\,dy \right) dx$$

Fubiniova věta: Je-li $M \subseteq \mathbb{R}^2$ „rozumná množina" (např. její hranice je dána spojitou funkcí a M omezená) a f je spojitá na M a omezená, pak se výše uvedené integrály existují a rovnají se.

Příklad:

Vypočítejme objem jehlanu dvojným integrálem.



rovnice přední stěny

$$z = 3 + ax + by$$

$$0 = 3 + 2a + 0$$

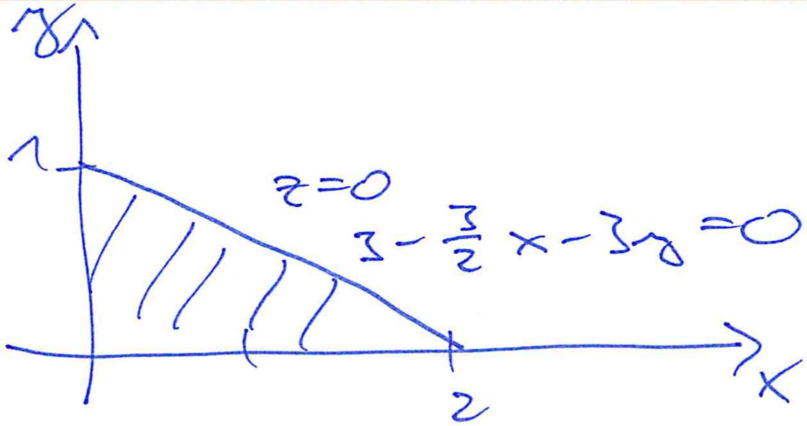
$$a = -\frac{3}{2}$$

$$0 = 3 + 0 + b$$

$$b = -3$$

$$z = 3 - \frac{3}{2}x - 3y$$

$$V = \iint_M 3 - \frac{3}{2}x - 3y \, dx\,dy$$

M: 

$z=0$

$3-\frac{3}{2}x-3y=0 \quad \dots \quad y=1-\frac{1}{2}x$

$x=2-2y$

dvojnásobné integrály:

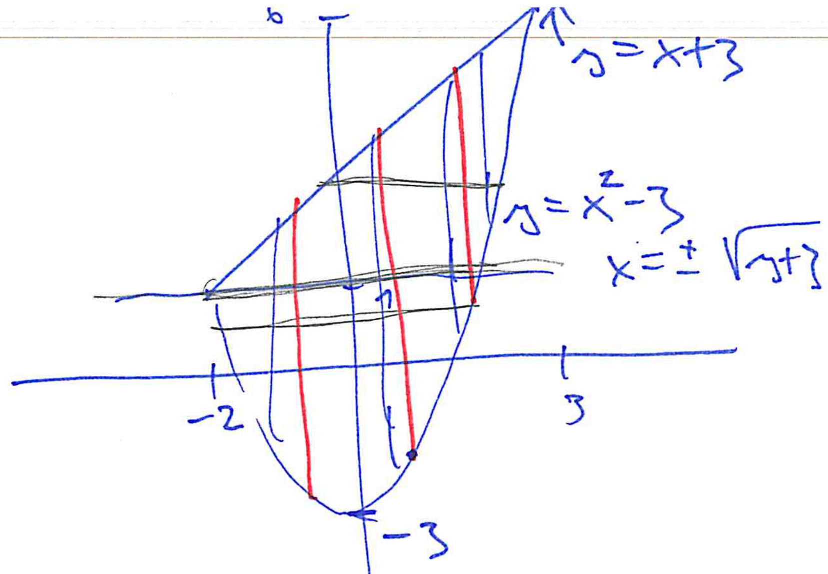
$$=\left[3x-\frac{3}{4}x^2-3xy\right]_0^{2-2y}=3(2-2y)-\frac{3}{4}(2-2y)^2-3(2-2y)y$$

$$\int_0^1\left(\boxed{\int_0^{2-2y} 3-\frac{3}{2}x-3y\,dx}\right)dy=\int_0^1 3(2-2y)-\frac{3}{4}(2-2y)^2-3(2-2y)y\,dy$$

$$=\dots=1$$

$$\int_0^2\left(\int_0^{1-\frac{1}{2}x} 3-\frac{3}{2}x-3y\,dy\right)dx=\dots=1$$

M:



$$\iint_M x^2\,dx\,dy$$

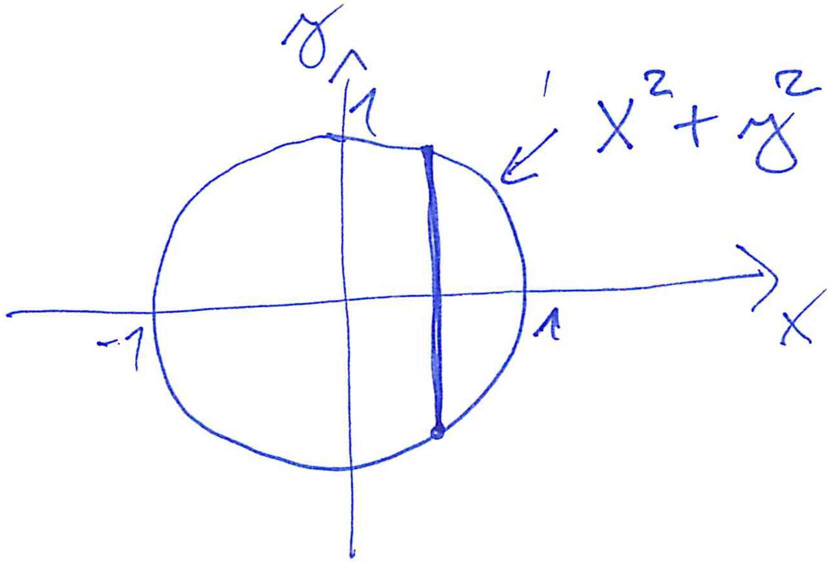
$$\int_{-2}^{3}\left(\int_{x^2-3}^{x+3} x^2\,dy\right)dx$$

$$= \left[x^2 y\right]_{x^2-3}^{x+3} = x^2(x+3) - x^2(x^2-3)$$

$$\int_{-3}^{1}\left(\int_{-\sqrt{y+3}}^{\sqrt{y+3}} x^2\,dx\right)dy + \int_{1}^{6}\left(\int_{y-3}^{\sqrt{y+3}} x^2\,dx\right)dy$$

$$\left[\frac{1}{3}x^3\right]_{-\sqrt{y+3}}^{\sqrt{y+3}}$$

Objem ~~koule~~ polokoule o poloměru $r=1$

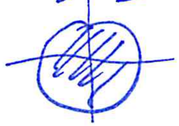


$x^2+y^2=1$ ← rovnice kružnice

rovnice sféry : $x^2+y^2+z^2=1$

$f(x,y)=\sqrt{1-x^2-y^2}$

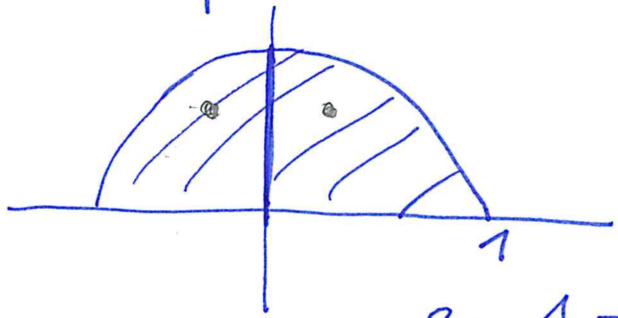
$$\iint_M \sqrt{1-x^2-y^2}\,dx\,dy$$

M : 

$$\int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,dy\right)dx$$

Těžiště rovinného obrazce M

půlkruh o poloměru 1



$$T = [x_T, y_T]$$

S ~~po~~ obsah obrazce

$$S = \iint_M 1 \, dx dy$$

$$S = \frac{1}{2}\pi$$

$$x_T = \frac{\iint_M x \, dx dy}{S}$$

$$x_T = 0$$

$$y_T = \frac{\iint_M y \, dx dy}{S}$$

$$y_T = ?$$

$$\int_{-1}^{1} \left(\boxed{\int_0^{\sqrt{1-x^2}} y \, dy} \right) dx$$

$$\boxed{\left[\frac{1}{2} y^2\right]_0^{\sqrt{1-x^2}}} = \frac{1}{2}(1-x^2) - 0$$

$$\int_{-1}^{1} \frac{1}{2}(1-x^2) \, dx = \left[\frac{1}{2}x - \frac{1}{6}x^3\right]_{-1}^{1}$$

$$= \frac{1}{2} - \frac{1}{6} - \left(-\frac{1}{2} + \frac{1}{6}\right) = \frac{2}{3}$$

$$y_T = \frac{\frac{2}{3}}{\frac{\pi}{2}} = \frac{4}{3\pi}$$

$$T = \left[0, \frac{4}{3\pi}\right]$$

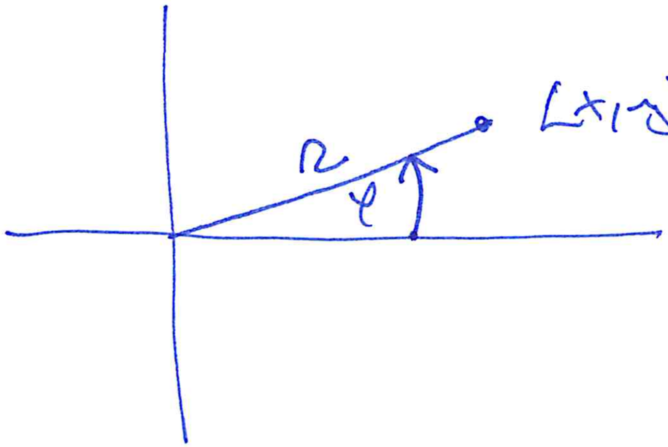
$$\int_{0}^{x_T} \quad (x_T = 0)$$

$$\int_{0}^{\sqrt{1-x^2}} x\,dy = \left[xy\right]_0^{\sqrt{1-x^2}} = x\sqrt{1-x^2} - 0$$

$$-\frac{1}{2}\int_{-1}^{1} -2x \sqrt{1-x^2}\,dx = \ldots = 0$$

substitueer $t = 1 - x^2$

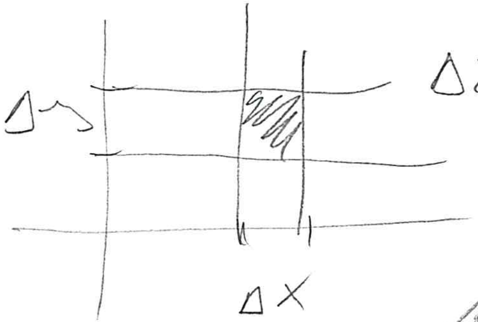
Polární souřadnice



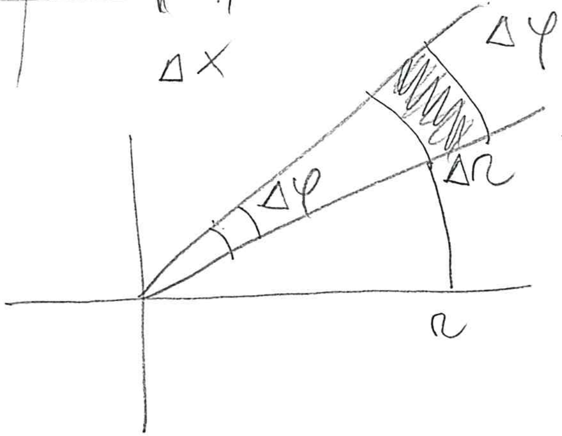
$r = \sqrt{x^2+y^2}$

$\mathrm{tg}\,\varphi = \dfrac{x}{y}$

Substituce kartézské → polární souřadnice
(ve dvojném integrálu)
$x, y \quad \to \quad r, \varphi$

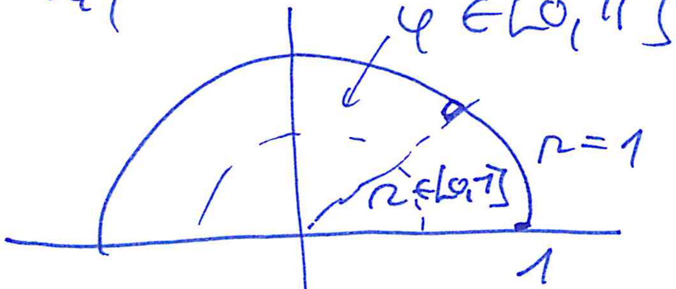


$\Delta x \cdot \Delta y$



$\Delta\varphi\left(r\Delta r + \frac{1}{2}(\Delta r)^2\right)$

$\sim r\,\Delta\varphi\,\Delta r$



dvojitý

$$S = \int_0^1 \left(\int_0^{\pi} 1 \cdot r \, d\varphi \right) dr$$

$[r\varphi]_0^{\pi} = \pi r \qquad \left[\frac{1}{2}\pi r^2\right]_0^1 = \frac{1}{2}\pi$

Objem polokoule

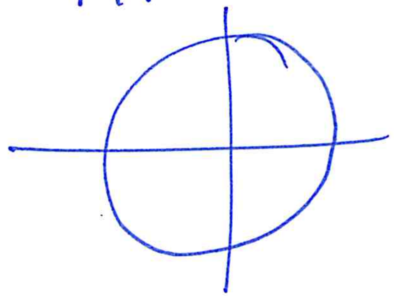
$$V = \iint \sqrt{1-x^2-y^2}\, dx\,dy$$

$M:$ 

$x = r\cos\varphi$

$y = r\sin\varphi$

$M:$



$r \in [0,1]$

$\varphi \in [0,2\pi]$

$$\sqrt{1-x^2-y^2} = \sqrt{1-r^2\cos^2\varphi - r^2\sin^2\varphi} = \sqrt{1-r^2}$$

$$\iint \sqrt{1-r^2}\, \underbrace{r\,dr\,d\varphi}_{dx\,dy} = \int_0^1 \left(\boxed{\int_0^{2\pi} r\sqrt{1-r^2}\, d\varphi} \right) dr$$

$$\int_0^1 2\pi r\sqrt{1-r^2}\, dr = \frac{2}{3}\pi$$

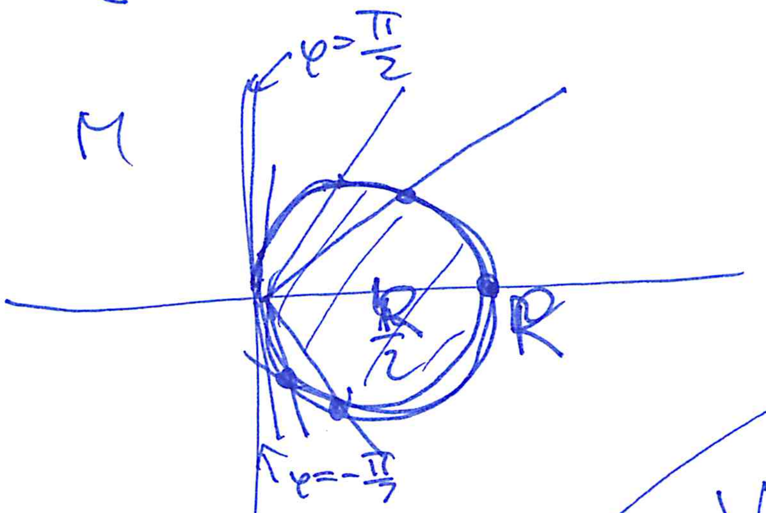
$t = 1-r^2$

$$M^* = \{[r,\varphi] : r\in[0,1],\ \varphi\in[0,2\pi]\} =$$

$$= [0,1]\times[0,2\pi]$$

Vivianiho křivka

objem tělesa



$r\cos\varphi$ $\quad$ $r\sin\varphi$

$$\left(x-\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2$$

$$V=\iint_M \sqrt{R^2-x^2-y^2}\,dx\,dy$$

$\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$r\in[0, ?]$

pro $\varphi=0$ je $r=R$

$\left.\begin{array}{l}\varphi=-\frac{\pi}{2}\\ \varphi=\frac{\pi}{2}\end{array}\right\}\ r=0$