actually produces individual photons one at a time, it should be mentioned that such devices can now be constructed (see Grangier *et al.* 1986).

As a final comment, I should remark that it is not necessary for a measuring device to be so dramatic an object as the bomb in this discussion. Indeed, such a 'device' need not actually signal its reception or non-reception of the photon to the outside world at all. A slightly wobbly mirror would, by itself, do as a measuring device, if it were light enough that it would move significantly as a result of the photon's impact and consequently disperse this motion as friction. The mere fact that the mirror is wobbly (say, the lower right-hand one, as before) will allow the detector at G to receive the photon, even if the mirror does *not* actually wobble, thus indicating that the photon went the other way. It is the mere *potentiality* for it to wobble that allows the photon to reach G! Even the obstruction, referred to in the previous section, plays an extremely similar role. It serves, in effect, to 'measure' the presence of the photon somewhere along its track as described by the successive states $|B\rangle$ and $|D\rangle$. A failure of it to receive the photon, when it is capable of receiving it, counts just as much as a 'measurement' as actually receiving it would.

Measurements of this negative and non-invasive kind are called *null* (or interaction-free) measurements, see Dicke (1981), and they have a considerable theoretical (and perhaps, eventually, even practical) importance. There are experiments to test directly the predictions of quantum theory in such situations. In particular, Kwiat, Weinfurter, and Zeilinger have recently performed an experiment of the *precise* type that is involved in the Elitzur–Vaidman bomb-testing problem! As we have now become used to accepting, the expectations of quantum theory have been completely confirmed. Null measurements are indeed among the profound Z-mysteries of quantum theory.

5.10 Quantum theory of spin; the Riemann sphere

In order to address the second of my two introductory quantum puzzles, it will be necessary to look into the structure of quantum theory in a little more detail. Recall that my dodecahedron, and also that of my colleague, had an atom of spin $\frac{3}{2}$ at its centre. What *is* spin, and what is its particular importance for quantum theory?

Spin is an intrinsic property of particles. It is basically the same physical concept as the spinning—or *angular momentum*—of a classical object, such as a golf ball, or cricket ball, or the whole earth. However, there is the (minor) difference that for such large objects, by far the major contribution to its angular momentum comes from the orbiting motions of all its particles about one another, whereas for a single particle, spin is a property that is intrinsic to the particle itself. In fact, the spin of a fundamental particle has the curious feature that its *magnitude* always has the *same* value, although the direction of its spin axis can vary—though this 'axis', also, behaves in a very odd way that

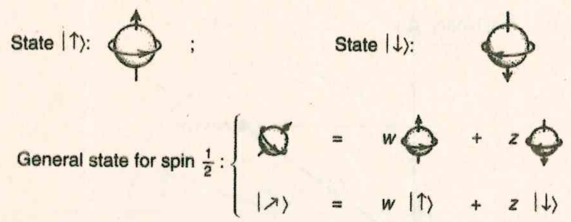

Fig. 5.15. For a particle of spin $\frac{1}{2}$ (such as an electron, proton, or neutron), all spin states are complex superpositions of the two states 'spin up' and 'spin down'.

bears little relation, in general, to what can happen classically. The magnitude of the spin is described in terms of the basic quantum-mechanical unit $\hbar$, which is Dirac's symbol for Planck's constant h, divided by 2π. The measure of spin of a particle is always a (non-negative) integer or half-integer multiple of $\hbar$, namely $0, \frac{1}{2}\hbar, \hbar, \frac{3}{2}\hbar, 2\hbar$, etc. We refer to such particles as having spin 0, spin $\frac{1}{2}$, spin 1, spin $\frac{3}{2}$, spin 2, etc., respectively.

Let us start by considering the simplest case (apart from spin 0, which is *too* simple, there being just one, spherically symmetrical, state of spin in this case), namely the case of spin $\frac{1}{2}$, such as that of an electron or a nucleon (a proton or neutron). For spin $\frac{1}{2}$, all states of spin are linear superpositions of just *two* states, say the state of right-handed spin about the *upward* vertical, written $|\uparrow\rangle$, or right-handed spin about the *downward* vertical, written $|\downarrow\rangle$ (see Fig. 5.15). The general state of spin would now be some complex-number combination $|\psi\rangle = w|\uparrow\rangle + z|\downarrow\rangle$. In fact it turns out that each such combination represents the state of the particle's spin (of magnitude $\frac{1}{2}\hbar$) being about some specific direction determined by the ratio of the two complex numbers w and z. There is nothing special about the particular choice of the two states $|\uparrow\rangle$ and $|\downarrow\rangle$. All the various combinations of these two states are just as clear-cut as states of spin as are the two original ones.

Let us see how this relationship can be made more explicit and geometrical. This will help us to appreciate that the complex-number weighting factors w and z are not quite such abstract things as they may have seemed to be, so far. In fact they have a clear relationship with the geometry of space. (I imagine that such geometric realizations might have pleased Cardano, and perhaps helped him with his 'mental tortures'—though quantum theory itself provides us with new mental tortures!)

It will be helpful, first, to consider the now-standard representation of complex numbers as points on a plane. (This plane is variously called the Argand plane, the Gauss plane, the Wessel plane, or just the *complex* plane.) The idea is simply to represent the complex number $z = x + iy$, where x and y are real numbers, by the point in the plane whose ordinary Cartesian coordinates are (x, y), with respect to some chosen Cartesian axes (see

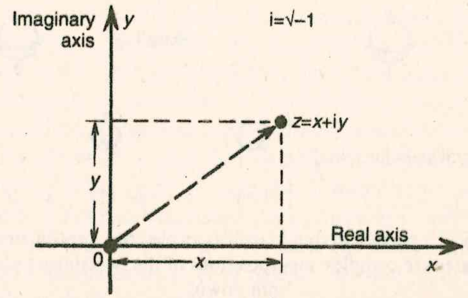

Fig. 5.16. The representation of a complex number in the (Wessel–Argand–Gauss) complex plane.

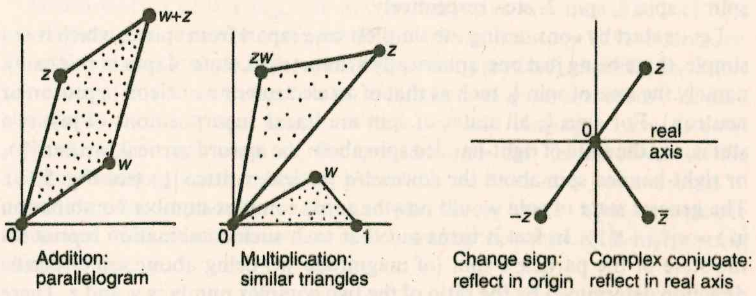

Fig. 5.17. The geometrical descriptions of the basic operations with complex numbers.

Fig. 5.16). Thus, for example, the four complex numbers 1, 1 + i, i, and 0 form the vertices of a square. There are simple geometrical rules for the sum and the product of two complex numbers (Fig. 5.17). Taking the negative $-z$ of a complex number z is represented by reflection in the origin; taking the complex conjugate $\bar{z}$, of z, by reflection in the x-axis.

The modulus of a complex number is the distance from the origin of the point representing it; the squared modulus is thus the square of this number. The *unit circle* is the locus of points that are of unit distance from the origin (Fig. 5.18), these representing the complex numbers of *unit modulus*, sometimes called *pure phases*, having the form

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

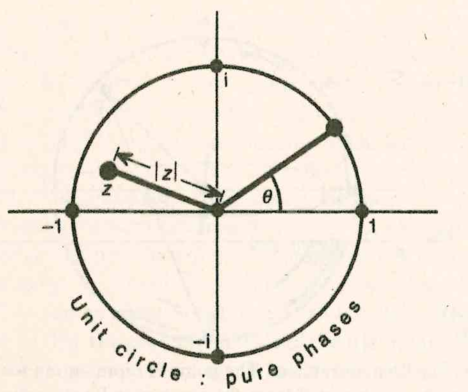

Fig. 5.18. The unit circle consists of complex numbers $z = e^{i\theta}$ with θ real, i.e. $|z| = 1$.

where θ is real and measures the angle that the line joining the origin to the point representing this complex number makes with the x-axis.*

Now let us see how to represent *ratios* of pairs of complex numbers. In the above discussion, I indicated that a state is not physically changed if it is mulplied, overall, by a non-zero complex number (recall, for example, that $-2|F\rangle$ was to be considered as being physically the same state as $|F\rangle$). Thus, generally, $|\psi\rangle$ is physically the same as $u|\psi\rangle$, for any non-zero complex number u. Applied to the state

$$|\psi\rangle = w|\uparrow\rangle + z|\downarrow\rangle,$$

We see that if we multiply both of w and z by the same non-zero complex number u, we do not change the physical situation that is represented by the state. It is the different *ratios* $z{:}w$ of the two complex numbers w and z that provide the different physically distinct spin states ($uz{:}uw$ being the same as $z{:}w$ if $u \neq 0$).

How do we geometrically represent a complex ratio? The essential difference between a complex ratio and just a plain complex number is that *infinity* (denoted by the symbol '∞') is also allowed as a ratio, in addition to all the finite complex numbers. Thus, if we think of the ratio $z{:}w$ to be represented, in general, by the single complex number z/w, we have trouble when $w = 0$. To cover this possibility, we simply use the symbol ∞ for z/w in the case when $w = 0$. This occurs when we consider the particular state 'spin down':

*The real number e is the 'base of natural logarithms': e = 2.718 281 828 5. . .; the expression e^z is, indeed 'e raised to the zth power', and we have

$$e^z = 1 + z + \frac{z^2}{1\times 2} + \frac{z^3}{1\times 2\times 3} + \frac{z^4}{1\times 2\times 3\times 4} + \ldots$$

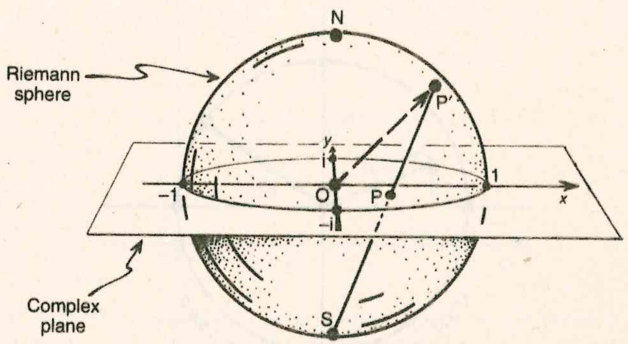

Fig. 5.19. The Riemann sphere. The point P, representing $p = z/w$ on the complex plane, is projected from the south pole S to a point, P′ on the sphere. The direction OP, out from the sphere's centre O, is the direction of spin for the general spin 1/2 state of Fig. 5.15.

$|\psi\rangle = z|\downarrow\rangle = 0|\uparrow\rangle + z|\downarrow\rangle$. Recall that we are not allowed to have *both* $w = 0$ and $z = 0$, but $w = 0$ by itself is perfectly allowable. (We could use w/z instead, to represent this ratio of we prefer; then we need ∞ to cover the case $z = 0$, which gives the particular state 'spin up'. It does not matter which description we use.)

The way to represent the space of all possible complex ratios is to use a *sphere*, referred to as the *Riemann sphere*. The points of the Riemann sphere represent complex numbers or ∞. We can picture the Riemann sphere as a sphere of unit radius whose equatorial plane is the complex plane and whose centre is the origin (zero) of that plane. The actual equator of this sphere will be identified with the unit circle in the complex plane (see Fig. 5.19). Now, to represent a particular complex ratio, say $z{:}w$, we mark the point P on the complex plane that represents the complex number $p = z/w$ (supposing, for the moment, that $w \neq 0$), and then we project P, on the plane, to a point P′ on the sphere from the *south pole* S. That is to say, we take the straight line from S to P and mark the point P′ on the sphere as the point where this line meets it (apart from at S itself). This mapping between points on the sphere and points on the plane is called *stereographic projection*. To see that it is reasonable for the south pole S itself to represent ∞, we imagine a point P in the plane that moves off to a very large distance; then we find that the point P′ that corresponds to it approaches the south pole S very closely, reaching S in the limit as P goes to infinity.

The Riemann sphere plays a fundamental role in the quantum picture of two-state systems. This role is not always evident explicitly, but the Riemann sphere is always there, behind the scenes. It describes, in an abstract geometrical way, the space of physically distinguishable states that can be built

up, by quantum linear superposition, from any two distinct quantum states. For example, the two states might be two possible locations for a photon, say $|B\rangle$ and $|C\rangle$. The general linear combination would have the form $w|B\rangle + z|C\rangle$. Although in §5.7 we made explicit use only of the particular case $|B\rangle + i|C\rangle$, as a result of the reflection/transmission at half-silvered mirror, the other combinations would not be hard to achieve. All that would be needed would be to vary the amount of silvering on the mirror and to introduce a segment of refracting medium in the path of one of the emerging beams. In this way, one could build up a complete Riemann sphere's worth of possible alternative states, given by all the various physical situations of the form $w|B\rangle + z|C\rangle$, that can be constructed from the two alternatives $|B\rangle$ and $|C\rangle$.

In cases such as this, the geometrical role of the Riemann sphere is not at all an apparent one. However, there are other types of situation in which the Riemann sphere's role is geometrically manifest. The clearest example of this occurs with the spin states of a particle of spin $\frac{1}{2}$, such as an electron or proton. The general state can be represented as a combination

$$|\psi\rangle = w|\uparrow\rangle + z|\downarrow\rangle,$$

and it turns out (choosing $|\uparrow\rangle$ and $|\downarrow\rangle$ appropriately from the proportionality class of physically equivalent possibilities) that this $|\psi\rangle$ represents the state of spin, of magnitude $\frac{1}{2}\hbar$, which is right handed about the axis that points in the direction of the very point on the Riemann sphere representing the ratio z/w. Thus every direction in space plays a role as a possible spin direction for any particle of spin $\frac{1}{2}$. Even though most states are represented, initially, as being 'mysterious complex-number-weighted combinations of alternatives' (the alternatives being $|\uparrow\rangle$ and $|\downarrow\rangle$), we see that these combinations are no more and no less mysterious than the two original ones, $|\uparrow\rangle$ and $|\downarrow\rangle$, that we started out with. Each is just as physically real as any of the others.

What about states of higher spin? It turns out that things get a little more complicated—and *more* mysterious! The general description that I shall give is not a very well-known one to today's physicists, although it was pointed out in 1932 by Ettore Majorana (a brilliant Italian physicist who disappeared at the age of 31 on a ship entering the Bay of Naples under circumstances that have never been fully explained).

Let us consider, first, what *is* very familiar to physicists. Suppose that we have an atom (or particle) of spin $\frac{1}{2}n$. Again, we can choose the upward direction to start with, and ask the question as to 'how much' of the atom's spin is actually oriented in (i.e. right-handed about) that direction. There is a standard piece of apparatus, known as a Stern–Gerlach apparatus, which achieves such measurements by use of an inhomogeneous magnetic field. What happens is that there are just $n+1$ different possible outcomes, which can be distinguished by the fact that the atom is found to lie in just one of $n+1$ different possible beams. See Fig. 5.20. The amount of the spin that lies in the

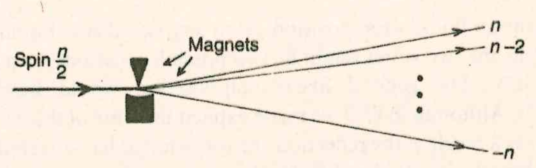

Fig. 5.20. The Stern–Gerlach measurement. There are $n+1$ possible outcomes, for spin $\frac{1}{2}n$, depending upon how much of the spin is found to be in the measured direction.

chosen direction is determined by the particular beam in which the atom is found to lie. When measured in units of $\frac{1}{2}\hbar$ the amount of spin in this direction turns out to have one of the values $n, n-2, n-4, \ldots, 2-n, -n$. Thus the different possible states of spin, for an atom of spin $\frac{1}{2}n$, are just the complex-number superpositions of these possibilities. I shall denote the various different possible results of a Stern–Gerlach measurement for spin $n+1$, when the field direction in the apparatus is the upward vertical, as

$$|\uparrow\uparrow\uparrow\ldots\uparrow\rangle, |\downarrow\uparrow\uparrow\ldots\uparrow\rangle, |\downarrow\downarrow\uparrow\ldots\uparrow\rangle, \ldots, |\downarrow\downarrow\downarrow\ldots\downarrow\rangle,$$

corresponding to the respective spin values $n, n-2, n-4, \ldots, 2-n, -n$ in that direction, where in each case there are exactly n arrows in all. We can think of each upward arrow as providing an amount $\frac{1}{2}\hbar$ of spin in the upward direction and each downward arrow as providing $\frac{1}{2}\hbar$ in the downward direction. Adding these values, we get the total amount of spin, in each case, obtained in a (Stern–Gerlach) spin measurement oriented in the up/down direction.

The general superposition of these is given by a complex combination

$$z_0|\uparrow\uparrow\uparrow\ldots\uparrow\rangle + z_1|\downarrow\uparrow\uparrow\ldots\uparrow\rangle + z_2|\downarrow\downarrow\uparrow\ldots\uparrow\rangle + \ldots + z_n|\downarrow\downarrow\downarrow\ldots\downarrow\rangle,$$

where the complex numbers $z_0, z_1, z_2, \ldots, z_n$ are not all zero. Can we represent such a state in terms of single directions of spin which are not simply 'up' or 'down'? What Majorana in effect showed was that this is indeed possible, but we must allow that the various arrows might point in quite independent directions; there is no need for them to be aligned in one pair of opposite directions, as would be the case in the result of a Stern–Gerlach measurement. Thus, we represent the general state of spin $\frac{1}{2}n$ as a collection of n independent such 'arrow directions'; we may think of these as being given by n points on the Riemann sphere, where each arrow direction is the direction out from the centre of the sphere to the relevant point on the sphere (Fig. 5.21). It is important to make clear that this is an *unordered* collection of points (or of arrow directions). Thus, there is no significance to be assigned to any ordering of the points into first, second, third, etc.

This is a very odd picture of spin, if we are to try to think of quantum-mechanical spin as the same phenomenon as the ordinary concept of spin that is familiar at the classical level. The spin of a classical object like a golf ball has

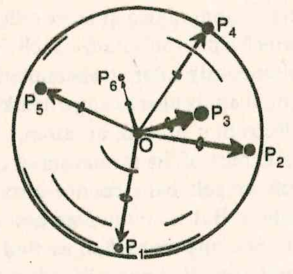

Fig. 5.21. The Majorana description of the general state of spin $\frac{1}{2}n$ is as an unordered set of n points $P_1, P_2, \ldots P_n$ on the Riemann sphere, where each point may be thought of as an element of spin $\frac{1}{2}$ directed outwards from the centre to the point in question.

a well-defined axis about which the object actually spins, whereas it appears that a quantum-level object is allowed to spin all at once about all kinds of axes pointing in many different directions. If we try to think that a classical object is really just the same as a quantum object, except that it is 'big' in some sense, then we seem to be presented with a paradox. The larger the magnitude of the spin, the more directions there are to be involved. Why, indeed, do classical objects not spin in many different directions all at once? This is an example of an **X**-mystery of quantum theory. Something comes to intervene (at an unspecified level), and we find that most types of quantum state do not arise (or, at least, almost never arise) at the classical level of phenomena that we can actually perceive. In the case of spin, what we find is that the only states that significantly persist at the classical level are those in which the arrow directions are mainly clustered about one particular direction: the spin direction (axis) of the classically spinning object.

There is something called the 'correspondence principle' in quantum theory that asserts, in effect, that when physical quantities (such as the magnitude of spin) get large, then it is *possible* for the system to behave in a way that closely approximates classical behaviour (such as with the state where the arrows roughly point all in the same direction). However, this principle does not tell us how such states can arise solely by action of the Schrödinger equation **U**. In fact 'classical states' almost never arise in this way. The classical-like states come about because of the action of a different procedure: state-vector reduction **R**.

5.11 Position and momentum of a particle

There is an even more clear-cut example of this sort of thing in the quantum-mechanical concept of *location* for a particle. We have seen that a particle's