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An Introduction to Complex Analysis

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Lecture 1
Complex Numbers I

We begin this lecture with the definition of complex numbers and then introduce basic operations-addition, subtraction, multiplication, and division of complex numbers. Next, we shall show how the complex numbers can be represented on the xy-plane. Finally, we shall define the modulus and conjugate of a complex number.

Throughout these lectures, the following well-known notations will be used:

$\mathbb{N} = \{1, 2, \cdots\}$, the set of all *natural numbers*;
$\mathbf{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$, the set of all *integers*;
$\mathbf{Q} = \{m/n : \ m, \ n \in \mathbf{Z}, \ n \neq 0\}$, the set of all *rational numbers*;
$\mathbb{R}$ = the set of all *real numbers.*

A *complex number* is an expression of the form $a + ib$, where a and $b \in \mathbb{R}$, and i (sometimes j) is just a symbol.

$\mathbf{C} = \{a + ib : \ a, \ b \in \mathbb{R}\}$, the set of all *complex numbers.*

It is clear that $\mathbb{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbb{R} \subset \mathbf{C}$.

For a complex number, $z = a + ib$, $\mathrm{Re}(z) = a$ is the *real part* of z, and $\mathrm{Im}(z) = b$ is the *imaginary part* of z. If $a = 0$, then z is said to be a *purely imaginary number.* Two complex numbers, z and w are equal; i.e., $z = w$, if and only if, $\mathrm{Re}(z) = \mathrm{Re}(w)$ and $\mathrm{Im}(z) = \mathrm{Im}(w)$. Clearly, $z = 0$ is the only number that is real as well as purely imaginary.

The following operations are defined on the complex number system:

(i). Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
(ii). Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$.
(iii). Multiplication: $(a + bi)(c + di) = (ac - bd) + (bc + ad)i$.

As in real number system, $0 = 0 + 0i$ is a complex number such that $z + 0 = z$. There is obviously a unique complex number 0 that possesses this property.

From (iii), it is clear that $i^2 = -1$, and hence, formally, $i = \sqrt{-1}$. Thus, except for zero, positive real numbers have real square roots, and negative real numbers have purely imaginary square roots.

R.P. Agarwal et al., *An Introduction to Complex Analysis*,
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For complex numbers z_1, z_2, z_3 we have the following easily verifiable properties:

(I). Commutativity of addition: $z_1 + z_2 = z_2 + z_1$.

(II). Commutativity of multiplication: $z_1 z_2 = z_2 z_1$.

(III). Associativity of addition: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

(IV). Associativity of multiplication: $z_1(z_2 z_3) = (z_1 z_2) z_3$.

(V). Distributive law: $(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$.

As an illustration, we shall show only (I). Let $z_1 = a_1 + b_1 i,\ z_2 = a_2 + b_2 i$ then

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + (b_1 + b_2)i \;=\; (a_2 + a_1) + (b_2 + b_1)i \\ &= (a_2 + b_2 i) + (a_1 + b_1 i) \;=\; z_2 + z_1. \end{aligned}$$

Clearly, $\mathbf{C}$ with addition and multiplication forms a field.

We also note that, for any integer k,

$$i^{4k} = 1, \quad i^{4k+1} = i, \quad i^{4k+2} = -1, \quad i^{4k+3} = -i.$$

The rule for division is derived as

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i, \quad c^2 + d^2 \neq 0.$$

Example 1.1. Find the quotient $\dfrac{(6+2i)-(1+3i)}{-1+i-2}$.

$$\begin{aligned} \frac{(6+2i)-(1+3i)}{-1+i-2} &= \frac{5-i}{-3+i} = \frac{(5-i)}{(-3+i)} \frac{(-3-i)}{(-3-i)} \\ &= \frac{-15-1-5i+3i}{9+1} = -\frac{8}{5} - \frac{1}{5} i. \end{aligned}$$

Geometrically, we can represent complex numbers as points in the xy-plane by associating to each complex number $a + bi$ the point (a, b) in the xy-plane (also known as an Argand diagram). The plane is referred to as the *complex plane*. The x-axis is called the *real axis*, and the y-axis is called the *imaginary axis*. The number $z = 0$ corresponds to the origin of the plane. This establishes a one-to-one correspondence between the set of all complex numbers and the set of all points in the complex plane.

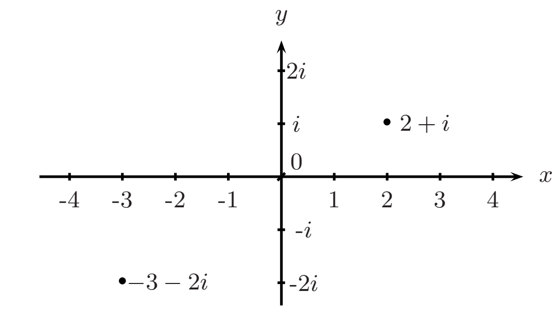

Figure 1.1

We can justify the above representation of complex numbers as follows: Let A be a point on the real axis such that $OA = a$. Since $i \cdot i\, a = i^2\, a = -a$, we can conclude that twice multiplication of the real number a by i amounts to the rotation of OA through two right angles to the position OA''. Thus, it naturally follows that the multiplication by i is equivalent to the rotation of OA through one right angle to the position OA'. Hence, if $y'Oy$ is a line perpendicular to the real axis $x'Ox$, then all imaginary numbers are represented by points on $y'Oy$.

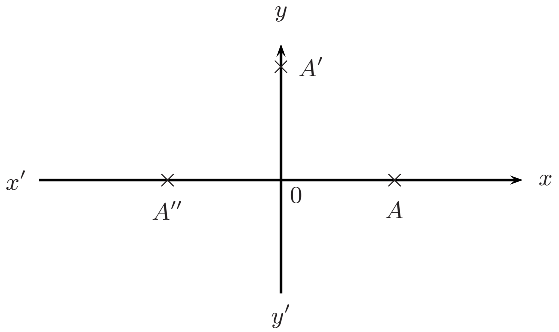

Figure 1.2

The *absolute value* or *modulus* of the number $z = a + ib$ is denoted by $|z|$ and given by $|z| = \sqrt{a^2 + b^2}$. Since $a \le |a| = \sqrt{a^2} \le \sqrt{a^2 + b^2}$ and $b \le |b| = \sqrt{b^2} \le \sqrt{a^2 + b^2}$, it follows that $\mathrm{Re}(z) \le |\mathrm{Re}(z)| \le |z|$ and $\mathrm{Im}(z) \le |\mathrm{Im}(z)| \le |z|$. Now, let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ then

$$|z_1 - z_2| \;=\; \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

Hence, $|z_1 - z_2|$ is just the distance between the points z_1 and z_2. This fact is useful in describing certain curves in the plane.

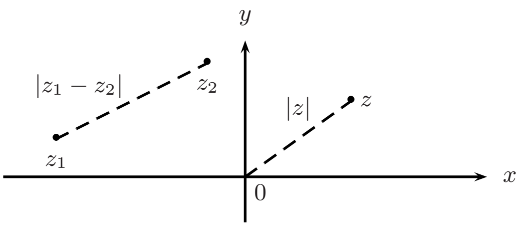

Figure 1.3

Example 1.2. The equation $|z-1+3i| = 2$ represents the circle whose center is $z_0 = 1-3i$ and radius is $R = 2$.

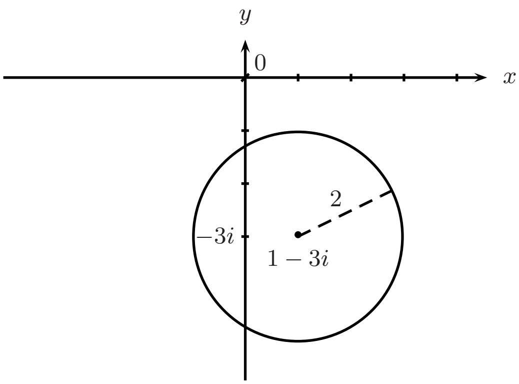

Figure 1.4

Example 1.3. The equation $|z+2| = |z-1|$ represents the perpendicular bisector of the line segment joining -2 and 1; i.e., the line $x = -1/2$.

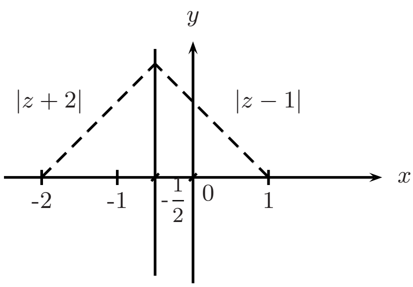

Figure 1.5

The *complex conjugate* of the number $z = a + bi$ is denoted by $\overline{z}$ and given by $\overline{z} = a - bi$. Geometrically, $\overline{z}$ is the reflection of the point z about the real axis.

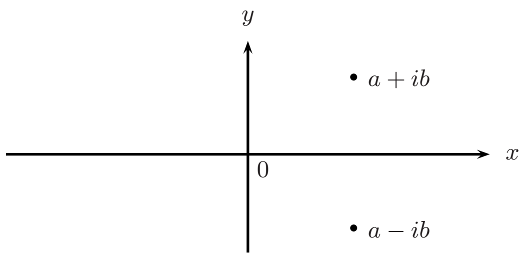

Figure 1.6

The following relations are immediate:

1. $|z_1 z_2| = |z_1||z_2|, \quad \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, \quad (z_2 \neq 0).$
2. $|z| \geq 0$, and $|z| = 0$, if and only if $z = 0$.
3. $z = \overline{z}$, if and only if $z \in \mathbb{R}$.
4. $z = -\overline{z}$, if and only if $z = bi$ for some $b \in \mathbb{R}$.
5. $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$.
6. $\overline{z_1 z_2} = (\overline{z}_1)(\overline{z}_2)$.
7. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}, \quad z_2 \neq 0.$
8. $\operatorname{Re}(z) = \dfrac{z + \overline{z}}{2}, \ \operatorname{Im}(z) = \dfrac{z - \overline{z}}{2i}.$
9. $\overline{\overline{z}} = z$.
10. $|z| = |\overline{z}|, \ z\overline{z} = |z|^2$.

As an illustration, we shall show only relation 6. Let $z_1 = a_1 + b_1 i, \ z_2 = a_2 + b_2 i$. Then

$$\begin{aligned}
\overline{z_1 z_2} &= \overline{(a_1 + b_1 i)(a_2 + b_2 i)} \\
&= \overline{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)} \\
&= (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2) \\
&= (a_1 - b_1 i)(a_2 - b_2 i) \ = \ (\overline{z}_1)(\overline{z}_2).
\end{aligned}$$

Lecture 2
Complex Numbers II

In this lecture, we shall first show that complex numbers can be viewed as two-dimensional vectors, which leads to the triangle inequality. Next, we shall express complex numbers in polar form, which helps in reducing the computation in tedious expressions.

For each point (number) z in the complex plane, we can associate a vector, namely the directed line segment from the origin to the point z; i.e., $z = a + bi \longleftrightarrow \overrightarrow{v} = (a, b)$. Thus, complex numbers can also be interpreted as two-dimensional *ordered pairs.* The length of the vector associated with z is $|z|$. If $z_1 = a_1 + b_1 i \ \longleftrightarrow\ \overrightarrow{v}_1 = (a_1, b_1)$ and $z_2 \ =\ a_2 + b_2 i \ \longleftrightarrow\ \overrightarrow{v}_2 = (a_2, b_2)$, then $z_1 + z_2 \longleftrightarrow \overrightarrow{v}_1 + \overrightarrow{v}_2$.

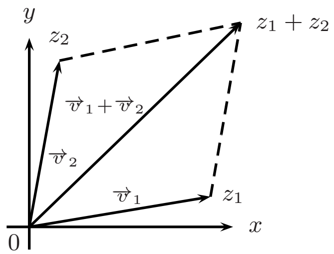

Figure 2.1

Using this correspondence and the fact that the length of any side of a triangle is less than or equal to the sum of the lengths of the two other sides, we have

$$|z_1 + z_2| \ \leq\ |z_1| + |z_2| \tag{2.1}$$

for any two complex numbers z_1 and z_2. This inequality also follows from

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \ =\ (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) \\
&= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2 \\
&= |z_1|^2 + (z_1\overline{z}_2 + \overline{z_1\overline{z}_2}) + |z_2|^2 \\
&= |z_1|^2 + 2\mathrm{Re}(z_1\overline{z}_2) + |z_2|^2 \\
&\leq |z_1|^2 + 2|z_1 z_2| + |z_2|^2 \ =\ (|z_1| + |z_2|)^2.
\end{aligned}$$

Applying the inequality (2.1) to the complex numbers $z_2 - z_1$ and z_1,

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we get

$$|z_2| \;=\; |z_2 - z_1 + z_1| \;\le\; |z_2 - z_1| + |z_1|,$$

and hence

$$|z_2| - |z_1| \;\le\; |z_2 - z_1|. \tag{2.2}$$

Similarly, we have

$$|z_1| - |z_2| \;\le\; |z_1 - z_2|. \tag{2.3}$$

Combining inequalities (2.2) and (2.3), we obtain

$$||z_1| - |z_2|| \;\le\; |z_1 - z_2|. \tag{2.4}$$

Each of the inequalities (2.1)-(2.4) will be called a *triangle inequality*. Inequality (2.4) tells us that the length of one side of a triangle is greater than or equal to the difference of the lengths of the two other sides. From (2.1) and an easy induction, we get the *generalized triangle inequality*

$$|z_1 + z_2 + \cdots + z_n| \;\le\; |z_1| + |z_2| + \cdots + |z_n|. \tag{2.5}$$

From the demonstration above, it is clear that, in (2.1), equality holds if and only if $\mathrm{Re}(z_1\overline{z}_2) = |z_1 z_2|$; i.e., $z_1\overline{z}_2$ is real and nonnegative. If $z_2 \neq 0$, then since $z_1\overline{z}_2 = z_1|z_2|^2/z_2$, this condition is equivalent to $z_1/z_2 \ge 0$. Now we shall show that equality holds in (2.5) if and only if the ratio of any two nonzero terms is positive. For this, if equality holds in (2.5), then, since

$$\begin{aligned}
|z_1 + z_2 + z_3 + \cdots + z_n| \;&=\; |(z_1 + z_2) + z_3 + \cdots + z_n| \\
&\le\; |z_1 + z_2| + |z_3| + \cdots + |z_n| \\
&\le\; |z_1| + |z_2| + |z_3| + \cdots + |z_n|,
\end{aligned}$$

we must have $|z_1 + z_2| = |z_1| + |z_2|$. But, this holds only when $z_1/z_2 \ge 0$, provided $z_2 \neq 0$. Since the numbering of the terms is arbitrary, the ratio of any two nonzero terms must be positive. Conversely, suppose that the ratio of any two nonzero terms is positive. Then, if $z_1 \neq 0$, we have

$$\begin{aligned}
|z_1 + z_2 + \cdots + z_n| \;&=\; |z_1|\left|1 + \frac{z_2}{z_1} + \cdots + \frac{z_n}{z_1}\right| \\
&=\; |z_1|\left(1 + \frac{z_2}{z_1} + \cdots + \frac{z_n}{z_1}\right) \\
&=\; |z_1|\left(1 + \frac{|z_2|}{|z_1|} + \cdots + \frac{|z_n|}{|z_1|}\right) \\
&=\; |z_1| + |z_2| + \cdots + |z_n|.
\end{aligned}$$

Example 2.1. If $|z| = 1$, then, from (2.5), it follows that

$$|z^2 + 2z + 6 + 8i| \;\le\; |z|^2 + 2|z| + |6 + 8i| \;=\; 1 + 2 + \sqrt{36 + 64} \;=\; 13.$$

Similarly, from (2.1) and (2.4), we find

$$2 \;\leq\; |z^2 - 3| \;\leq\; 4.$$

Note that the product of two complex numbers z_1 and z_2 is a new complex number that can be represented by a vector in the same plane as the vectors for z_1 and z_2. However, this product is neither the scalar (dot) nor the vector (cross) product used in ordinary vector analysis.

Now let $z = x + yi,\ r = |z| = \sqrt{x^2 + y^2}$, and θ be a number satisfying

$$\cos\theta \;=\; \frac{x}{r} \quad \text{and} \quad \sin\theta \;=\; \frac{y}{r}.$$

Then, z can be expressed in *polar* (*trigonometric*) *form* as

$$z \;=\; r(\cos\theta + i\sin\theta).$$

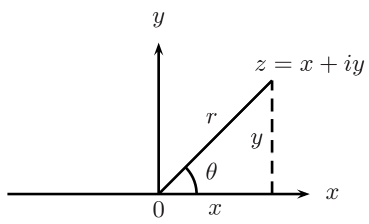

Figure 2.2

To find θ, we usually compute $\tan^{-1}(y/x)$ and adjust the quadrant problem by adding or subtracting π when appropriate. Recall that $\tan^{-1}(y/x) \in (-\pi/2, \pi/2)$.

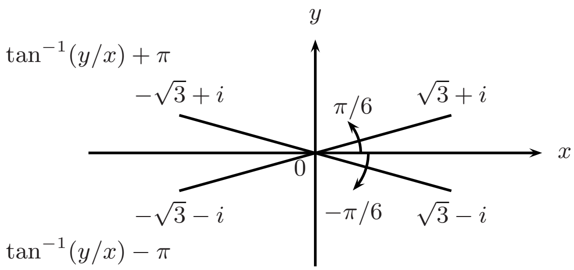

Figure 2.3

Example 2.2. Express $1-i$ in polar form. Here $r = \sqrt{2}$ and $\theta = -\pi/4$, and hence

$$1 - i \;=\; \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right].$$

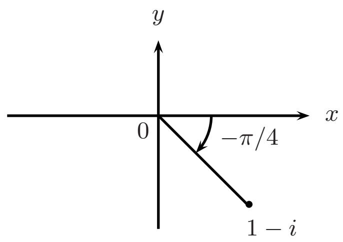

Figure 2.4

We observe that any one of the values $\theta = -(\pi/4) \pm 2n\pi,\ n = 0, 1, \cdots,$ can be used here. The number θ is called an *argument* of z, and we write $\theta = \arg z$. Geometrically, $\arg z$ denotes the angle measured in radians that the vector corresponds to z makes with the positive real axis. The argument of 0 is not defined. The pair $(r, \arg z)$ is called the *polar coordinates* of the complex number z.

The *principal value* of $\arg z$, denoted by $\operatorname{Arg} z$, is defined as that unique value of $\arg z$ such that $-\pi < \arg z \leq \pi$.

If we let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

Thus, $|z_1 z_2| = |z_1||z_2|,\ \arg(z_1 z_2) = \arg z_1 + \arg z_2$.

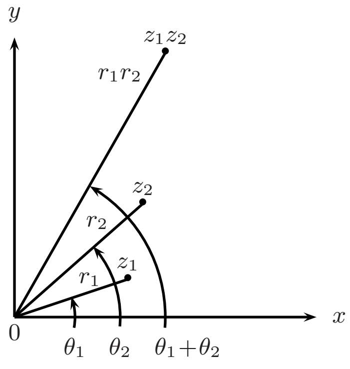

Figure 2.5

For the division, we have

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \\ \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|}, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \end{aligned}$$

Example 2.3. Write the quotient $\dfrac{1+i}{\sqrt{3}-i}$ in polar form. Since the polar forms of $1+i$ and $\sqrt{3}-i$ are

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad \sqrt{3}-i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right),$$

it follows that

$$\begin{aligned}\frac{1+i}{\sqrt{3}-i} &= \frac{\sqrt{2}}{2}\left\{\cos\left[\frac{\pi}{4} - \left(-\frac{\pi}{6}\right)\right] + i\sin\left[\frac{\pi}{4} - \left(-\frac{\pi}{6}\right)\right]\right\} \\ &= \frac{\sqrt{2}}{2}\left\{\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right\}.\end{aligned}$$

Recall that, geometrically, the point $\overline{z}$ is the reflection in the real axis of the point z. Hence, $\arg\overline{z} = -\arg z$.

Lecture 3
Complex Numbers III

In this lecture, we shall first show that every complex number can be written in exponential form, and then use this form to raise a rational power to a given complex number. We shall also extract roots of a complex number. Finally, we shall prove that complex numbers cannot be ordered.

If $z = x + iy$, then e^z is defined to be the complex number

$$e^z \;=\; e^x(\cos y + i\sin y). \tag{3.1}$$

This number e^z satisfies the usual algebraic properties of the exponential function. For example,

$$e^{z_1}e^{z_2} \;=\; e^{z_1+z_2} \quad \text{and} \quad \frac{e^{z_1}}{e^{z_2}} \;=\; e^{z_1-z_2}.$$

In fact, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then, in view of Lecture 2, we have

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2) \\ &= e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2)) \\ &= e^{(x_1+x_2)+i(y_1+y_2)} \;=\; e^{z_1+z_2}. \end{aligned}$$

In particular, for $z = iy$, the definition above gives one of the most important formulas of Euler

$$e^{iy} \;=\; \cos y + i\sin y, \tag{3.2}$$

which immediately leads to the following identities:

$$\cos y \;=\; \mathrm{Re}(e^{iy}) \;=\; \frac{e^{iy}+e^{-iy}}{2}, \quad \sin y \;=\; \mathrm{Im}(e^{iy}) \;=\; \frac{e^{iy}-e^{-iy}}{2i}.$$

When $y = \pi$, formula (3.2) reduces to the amazing equality $e^{\pi i} = -1$. In this relation, the transcendental number e comes from calculus, the transcendental number π comes from geometry, and i comes from algebra, and the combination $e^{\pi i}$ gives -1, the basic unit for generating the arithmetic system for counting numbers.

Using Euler's formula, we can express a complex number $z = r(\cos\theta + i\sin\theta)$ in exponential form; i.e.,

$$z \;=\; r(\cos\theta + i\sin\theta) \;=\; re^{i\theta}. \tag{3.3}$$

R.P. Agarwal et al., *An Introduction to Complex Analysis*,
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The rules for multiplying and dividing complex numbers in exponential form are given by

$$\begin{aligned} z_1 z_2 &= (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1+\theta_2)}, \\ \frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}. \end{aligned}$$

Finally, the complex conjugate of the complex number $z = re^{i\theta}$ is given by $\overline{z} = re^{-i\theta}$.

Example 3.1. Compute (1). $\dfrac{1+i}{\sqrt{3}-i}$ and (2). $(1+i)^{24}$.

(1). We have $1+i = \sqrt{2}e^{i\pi/4}$, $\sqrt{3}-i = 2e^{-i\pi/6}$, and therefore

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{i5\pi/12}.$$

(2). $(1+i)^{24} = (\sqrt{2}e^{i\pi/4})^{24} = 2^{12}e^{i6\pi} = 2^{12}$.

From the exponential representation of complex numbers, *De Moivre's formula*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n = 1, 2, \cdots, \tag{3.4}$$

follows immediately. In fact, we have

$$\begin{aligned} (\cos\theta + i\sin\theta)^n = (e^{i\theta})^n &= e^{i\theta}\cdot e^{i\theta}\cdots e^{i\theta} \\ &= e^{i\theta+i\theta+\cdots+i\theta} \\ &= e^{in\theta} = \cos n\theta + i\sin n\theta. \end{aligned}$$

From (3.4), it is immediate to deduce that

$$\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \frac{1+i\tan n\theta}{1-i\tan n\theta}.$$

Similarly, since

$$1+\sin\theta \pm i\cos\theta = 2\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right) \pm i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right],$$

it follows that

$$\left[\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right]^n = \cos\left(\frac{n\pi}{2}-n\theta\right) + i\sin\left(\frac{n\pi}{2}-n\theta\right).$$

Example 3.2. Express $\cos 3\theta$ in terms of $\cos\theta$. We have

$$\begin{aligned}\cos 3\theta &= \operatorname{Re}(\cos 3\theta + i\sin 3\theta) = \operatorname{Re}(\cos\theta + i\sin\theta)^3\\ &= \operatorname{Re}[\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(-\sin^2\theta) - i\sin^3\theta]\\ &= \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

Now, let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$. By using the multiplicative property of the exponential function, we get

$$z^n = r^n e^{in\theta} \tag{3.5}$$

for any positive integer n. If $n = -1, -2, \cdots$, we define z^n by $z^n = (z^{-1})^{-n}$. If $z = re^{i\theta}$, then $z^{-1} = e^{-i\theta}/r$. Hence,

$$z^n = (z^{-1})^{-n} = \left[\frac{1}{r}e^{i(-\theta)}\right]^{-n} = \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} = r^n e^{in\theta}.$$

Hence, formula (3.5) is also valid for negative integers n.

Now we shall see if (3.5) holds for $n = 1/m$. If we let

$$\xi = \sqrt[m]{r}e^{i\theta/m}, \tag{3.6}$$

then ξ certainly satisfies $\xi^m = z$. But it is well-known that the equation $\xi^m = z$ has more than one solution. To obtain all the mth roots of z, we must apply formula (3.5) to every polar representation of z. For example, let us find all the mth roots of unity. Since

$$1 = e^{2k\pi i}, \quad k = 0, \pm 1, \pm 2, \cdots,$$

applying formula (3.5) to every polar representation of 1, we see that the complex numbers

$$z = e^{(2k\pi i)/m}, \quad k = 0, \pm 1, \pm 2, \cdots,$$

are mth roots of unity. All these roots lie on the unit circle centered at the origin and are equally spaced around the circle every $2\pi/m$ radians.

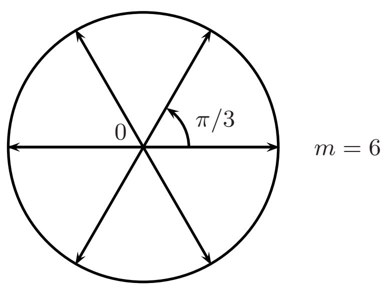

Figure 3.1

Hence, all of the distinct m roots of unity are obtained by writing

$$z \;=\; e^{(2k\pi i)/m}, \quad k = 0, 1, \cdots, m-1. \tag{3.7}$$

In the general case, the m distinct roots of a complex number $z = re^{i\theta}$ are given by

$$z^{1/m} \;=\; \sqrt[m]{r}e^{i(\theta+2k\pi)/m}, \quad k = 0, 1, \cdots, m-1.$$

Example 3.3. Find all the cube roots of $\sqrt{2}+i\sqrt{2}$. In polar form, we have $\sqrt{2}+i\sqrt{2} = 2e^{i\pi/4}$. Hence,

$$(\sqrt{2}+i\sqrt{2})^{1/3} \;=\; \sqrt[3]{2}e^{i\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2;$$

i.e.,

$$\sqrt[3]{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right),\ \sqrt[3]{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right),\ \sqrt[3]{2}\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right),$$

are the cube roots of $\sqrt{2}+i\sqrt{2}$.

Example 3.4. Solve the equation $(z+1)^5 = z^5$. We rewrite the equation as $\left(\dfrac{z+1}{z}\right)^5 = 1$. Hence,

$$\frac{z+1}{z} \;=\; e^{2k\pi i/5}, \quad k = 0, 1, 2, 3, 4,$$

or

$$z \;=\; \frac{1}{e^{2k\pi i/5}-1} \;=\; -\frac{1}{2}\left(1+i\cot\frac{\pi k}{5}\right), \quad k = 0, 1, 2, 3, 4.$$

Similarly, for any natural number n, the roots of the equation $(z+1)^n + z^n = 0$ are

$$z \;=\; -\frac{1}{2}\left(1+i\cot\frac{\pi+2k\pi}{n}\right), \quad k = 0, 1, \cdots, n-1.$$

We conclude this lecture by proving that complex numbers cannot be ordered. (Recall that the definition of the order relation denoted by $>$ in the real number system is based on the existence of a subset $\mathcal{P}$ (the positive reals) having the following properties: (i) For any number $\alpha \neq 0$, either α or $-\alpha$ (but not both) belongs to $\mathcal{P}$. (ii) If α and β belong to $\mathcal{P}$, so does $\alpha+\beta$. (iii) If α and β belong to $\mathcal{P}$, so does $\alpha\cdot\beta$. When such a set $\mathcal{P}$ exists, we write $\alpha > \beta$ if and only if $\alpha - \beta$ belongs to $\mathcal{P}$.) Indeed, suppose there is a nonempty subset $\mathcal{P}$ of the complex numbers satisfying (i), (ii), and (iii). Assume that $i \in \mathcal{P}$. Then, by (iii), $i^2 = -1 \in \mathcal{P}$ and $(-1)i = -i \in \mathcal{P}$. This violates (i). Similarly, (i) is violated by assuming $-i \in \mathcal{P}$. Therefore, the words positive and negative are never applied to complex numbers.

Problems

3.1. Express each of the following complex numbers in the form $x+iy$:
(a). $(\sqrt{2}-i)-i(1-\sqrt{2}i)$, (b). $(2-3i)(-2+i)$, (c). $(1-i)(2-i)(3-i)$,
(d). $\dfrac{4+3i}{3-4i}$, (e). $\dfrac{1+i}{i}+\dfrac{i}{1-i}$, (f). $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$,
(g). $(1+\sqrt{3}\,i)^{-10}$, (h). $(-1+i)^7$, (i). $(1-i)^4$.

3.2. Describe the following loci or regions:
(a). $|z-z_0| = |z-\overline{z}_0|$, where $\operatorname{Im} z_0 \neq 0$,
(b). $|z-z_0| = |z+\overline{z}_0|$, where $\operatorname{Re} z_0 \neq 0$,
(c). $|z-z_0| = |z-z_1|$, where $z_0 \neq z_1$,
(d). $|z-1| = 1$,
(e). $|z-2| = 2|z-2i|$,
(f). $\left|\dfrac{z-z_0}{z-z_1}\right| = c$, where $z_0 \neq z_1$ and $c \neq 1$,
(g). $0 < \operatorname{Im} z < 2\pi$,
(h). $\dfrac{\operatorname{Re} z}{|z-1|} > 1$, $\operatorname{Im} z < 3$,
(i). $|z-z_1| + |z-z_2| = 2a$,
(j). $az\overline{z} + kz + \overline{k}\overline{z} + d = 0$, $k \in \mathbf{C}$, $a, d \in \mathbb{R}$, and $|k|^2 > ad$.

3.3. Let α, $\beta \in \mathbf{C}$. Prove that

$$|\alpha+\beta|^2 + |\alpha-\beta|^2 \;=\; 2(|\alpha|^2 + |\beta|^2),$$

and deduce that

$$|\alpha + \sqrt{\alpha^2-\beta^2}| + |\alpha - \sqrt{\alpha^2-\beta^2}| \;=\; |\alpha+\beta| + |\alpha-\beta|.$$

3.4. Use the properties of conjugates to show that

(a). $(\overline{z})^4 = \overline{(z^4)}$, (b). $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2\overline{z}_3}$.

3.5. If $|z| = 1$, then show that

$$\left|\frac{az+b}{\overline{b}z+\overline{a}}\right| \;=\; 1$$

for all complex numbers a and b.

3.6. If $|z| = 2$, use the triangle inequality to show that

$$|\mathrm{Im}(1 - \overline{z} + z^2)| \le 7 \quad \text{and} \quad |z^4 - 4z^2 + 3| \ge 3.$$

3.7. Prove that if $|z| = 3$, then

$$\frac{5}{13} \le \left|\frac{2z-1}{4+z^2}\right| \le \frac{7}{5}.$$

3.8. Let z and w be such that $\overline{z}w \ne 1$, $|z| \le 1$, and $|w| \le 1$. Prove that

$$\left|\frac{z-w}{1-\overline{z}w}\right| \le 1.$$

Determine when equality holds.

3.9. (a). Prove that z is either real or purely imaginary if and only if $(\overline{z})^2 = z^2$.
(b). Prove that $\sqrt{2}|z| \ge |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$.

3.10. Show that there are complex numbers z satisfying $|z-a|+|z+a| = 2|b|$ if and only if $|a| \le |b|$. If this condition holds, find the largest and smallest values of $|z|$.

3.11. Let $z_1, z_2, \cdots, z_n$ and $w_1, w_2, \cdots, w_n$ be complex numbers. Establish *Lagrange's identity*

$$\left|\sum_{k=1}^n z_k w_k\right|^2 = \left(\sum_{k=1}^n |z_k|^2\right)\left(\sum_{k=1}^n |w_k|^2\right) - \sum_{k<\ell} |z_k\overline{w}_\ell - z_\ell\overline{w}_k|^2,$$

and deduce *Cauchy's inequality*

$$\left|\sum_{k=1}^n z_k w_k\right|^2 \le \left(\sum_{k=1}^n |z_k|^2\right)\left(\sum_{k=1}^n |w_k|^2\right).$$

3.12. Express the following in the form $r(\cos\theta + i\sin\theta)$, $-\pi < \theta \le \pi$:

(a). $\dfrac{(1-i)(\sqrt{3}+i)}{(1+i)(\sqrt{3}-i)}$, (b). $-8 + \dfrac{4}{i} + \dfrac{25}{3-4i}$.

3.13. Find the principal argument (Arg) for each of the following complex numbers:

(a). $5\left(\cos\dfrac{\pi}{8} - i\sin\dfrac{\pi}{8}\right)$, (b). $-3+\sqrt{3}i$, (c). $-\dfrac{2}{1+\sqrt{3}i}$, (d). $(\sqrt{3}-i)^6$.

3.14. Given $z_1 z_2 \neq 0$, prove that

$$\operatorname{Re} z_1 \overline{z}_2 \;=\; |z_1||z_2| \quad \text{if and only if} \quad \operatorname{Arg} z_1 \;=\; \operatorname{Arg} z_2.$$

Hence, show that

$$|z_1 + z_2| \;=\; |z_1| + |z_2| \quad \text{if and only if} \quad \operatorname{Arg} z_1 \;=\; \operatorname{Arg} z_2.$$

3.15. What is wrong in the following?

$$1 \;=\; \sqrt{1} \;=\; \sqrt{(-1)(-1)} \;=\; \sqrt{-1}\sqrt{-1} \;=\; i\,i \;=\; -1.$$

3.16. Show that

$$\frac{(1-i)^{49}\left(\cos\frac{\pi}{40} + i\sin\frac{\pi}{40}\right)^{10}}{(8i - 8\sqrt{3})^6} \;=\; -\sqrt{2}.$$

3.17. Let $z = re^{i\theta}$ and $w = Re^{i\phi}$, where $0 < r < R$. Show that

$$\operatorname{Re}\left(\frac{w+z}{w-z}\right) \;=\; \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}.$$

3.18. Solve the following equations:

(a). $z^2 = 2i$, (b). $z^2 = 1 - \sqrt{3}i$, (c). $z^4 = -16$, (d). $z^4 = -8 - 8\sqrt{3}i$.

3.19. For the root of unity $z = e^{2\pi i/m},\ m > 1$, show that

$$1 + z + z^2 + \cdots + z^{m-1} \;=\; 0.$$

3.20. Let a and b be two real constants and n be a positive integer. Prove that all roots of the equation

$$\left(\frac{1+iz}{1-iz}\right)^n \;=\; a + ib$$

are real if and only if $a^2 + b^2 = 1$.

3.21. A *quarternion* is an ordered pair of complex numbers; e.g., $((1,2),(3,4))$ and $(2+i, 1-i)$. The sum of quarternions (A, B) and (C, D) is defined as $(A + C, B + D)$. Thus, $((1,2),(3,4)) + ((5,6),(7,8)) = ((6,8),(10,12))$ and $(1-i, 4+i) + (7+2i, -5+i) = (8+i, -1+2i)$. Similarly, the scalar multiplication by a complex number A of a quaternion (B, C) is defined by the quadternion (AB, AC). Show that the addition and scalar multiplication of quaternions satisfy all the properties of addition and multiplication of real numbers.

3.22. Observe that:

(a). If $x = 0$ and $y > 0\,(y < 0)$, then $\operatorname{Arg} z = \pi/2\,(-\pi/2)$.

(b). If $x > 0$, then $\operatorname{Arg} z = \tan^{-1}(y/x) \in (-\pi/2, \pi/2)$.

(c). If $x < 0$ and $y > 0\,(y < 0)$, then $\operatorname{Arg} z = \tan^{-1}(y/x) + \pi$ $(\tan^{-1}(y/x) - \pi)$.

(d). $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2m\pi$ for some integer m. This m is uniquely chosen so that the LHS $\in (-\pi, \pi]$. In particular, let $z_1 = -1,\ z_2 = -1$, so that $\operatorname{Arg} z_1 = \operatorname{Arg} z_2 = \pi$ and $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(1) = 0$. Thus the relation holds with $m = -1$.

(e). $\operatorname{Arg}(z_1/z_2) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 + 2m\pi$ for some integer m. This m is uniquely chosen so that the LHS $\in (-\pi, \pi]$.

Answers or Hints

3.1. (a). $-2i$, (b). $-1 + 8i$, (c). $-10i$, (d). i, (e). $(1 - i)/2$, (f). $-2/5$, (g). $2^{-11}(-1 + \sqrt{3}i)$, (h). $-8(1 + i)$, (i). -4.
3.2. (a). Real axis, (b). imaginary axis, (c). perpendicular bisector (passing through the origin) of the line segment joining the points z_0 and z_1, (d). circle center $z = 1$, radius 1; i.e., $(x - 1)^2 + y^2 = 1$, (e). circle center $(-2/3, 8/3)$, radius $\sqrt{32}/3$, (f). circle, (g). $0 < y < 2\pi$, infinite strip, (h). region interior to parabola $y^2 = 2(x - 1/2)$ but below the line $y = 3$, (i). ellipse with foci at z_1, z_2 and major axis $2a$ (j). circle.
3.3. Use $|z|^2 = z\overline{z}$.
3.4. (a). $\overline{z^4} = \overline{zzzz} = \overline{z}\,\overline{z}\,\overline{z}\,\overline{z} = (\overline{z})^4$, (b). $\overline{\left(\frac{z_1}{z_2 z_3}\right)} = \frac{\overline{z}_1}{\overline{z_2 z_3}} = \frac{\overline{z}_1}{\overline{z}_2 \overline{z}_3}$.
3.5. If $|z| = 1$, then $\overline{z} = z^{-1}$.
3.6. $|\operatorname{Im}(1 - \overline{z} + z^2)| \le |1 - \overline{z} + z^2| \le |1| + |\overline{z}| + |z^2| \le 7$, $|z^4 - 4z^2 + 3| = |z^2 - 3||z^2 - 1| \ge (|z^2| - 3)(|z^2| - 1)$.
3.7. We have

$$\left|\frac{2z - 1}{4 + z^2}\right| \le \frac{2|z| + 1}{|4 - |z|^2|} = \frac{2 \cdot 3 + 1}{|4 - 3^2|} = \frac{7}{5}$$

and

$$\left|\frac{2z - 1}{4 + z^2}\right| \ge \frac{|2|z| - 1|}{|4 + |z|^2|} = \frac{2 \cdot 3 - 1}{4 + 3^2} = \frac{5}{13}.$$

3.8. We shall prove that $|1 - \overline{z}w| \ge |z - w|$. We have $|1 - \overline{z}w|^2 - |z - w|^2 = (1 - \overline{z}w)(1 - z\overline{w}) - (z - w)(\overline{z} - \overline{w}) = 1 - z\overline{w} - \overline{z}w + \overline{z}wz\overline{w} - z\overline{z} + z\overline{w} + w\overline{z} - w\overline{w} = 1 - |z|^2 - |w|^2 + |z|^2|w|^2 = (1 - |z|^2)(1 - |w|^2) \ge 0$ since $|z| \le 1$ and $|w| \le 1$. Equality holds when $|z| = |w| = 1$.
3.9. (a). $(\overline{z})^2 = z^2$ iff $z^2 - (\overline{z})^2 = 0$ iff $(z + \overline{z})(z - \overline{z}) = 0$ iff either $2\operatorname{Re}(z) = z + \overline{z} = 0$ or $2i\operatorname{Im}(z) = z - \overline{z} = 0$ iff z is purely imaginary or z is real. (b). Write $z = x + iy$. Consider $2|z|^2 - (|\operatorname{Re} z| + |\operatorname{Im} z|)^2 = 2(x^2 + y^2) - (|x| + |y|)^2 = 2x^2 + 2y^2 - (x^2 + y^2 + 2|x|y|) = x^2 + y^2 - 2|x||y| = (|x| - |y|)^2 \ge 0$.
3.10. Use the triangle inequality.

3.11. We have

$$\begin{aligned}\left|\sum_{k=1}^{n} z_k w_k\right|^2 &= \left(\sum_{k=1}^{n} z_k w_k\right)\left(\sum_{\ell=1}^{n} \overline{z}_\ell \overline{w}_\ell\right) = \sum_{k=1}^{n} |z_k|^2|w_k|^2 + \sum_{k\neq\ell} z_k w_k \overline{z}_\ell \overline{w}_\ell \\ &= \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) - \sum_{k\neq\ell} |z_k|^2|w_\ell|^2 + \sum_{k\neq\ell} z_k w_k \overline{z}_\ell \overline{w}_\ell \\ &= \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) - \sum_{k<\ell} |z_k \overline{w}_\ell - z_\ell \overline{w}_k|^2.\end{aligned}$$

3.12. (a). $\cos(-\pi/6) + i\sin(-\pi/6)$, (b). $5(\cos\pi + i\sin\pi)$.
3.13. (a). $-\pi/8$, (b). $5\pi/6$, (c). $2\pi/3$, (d). π.
3.14. Let $z_1 = r_1 e^{i\theta_1},\ z_2 = r_2 e^{i\theta_2}$. Then, $z_1\overline{z}_2 = r_1 r_2 e^{i(\theta_1-\theta_2)}$. $\mathrm{Re}(z_1\overline{z}_2) = r_1 r_2\cos(\theta_1-\theta_2) = r_1 r_2$ if and only if $\theta_1 - \theta_2 = 2k\pi,\ k \in \mathbf{Z}$. Thus, if and only if $\mathrm{Arg}\, z_1$-$\mathrm{Arg}\, z_2 = 2k\pi,\ k \in \mathbf{Z}$. But for $-\pi < \mathrm{Arg}\, z_1,\ \mathrm{Arg}\, z_2 \le \pi$, the only possibility is $\mathrm{Arg}\, z_1 = \mathrm{Arg}\, z_2$. Conversely, if $\mathrm{Arg}\, z_1 = \mathrm{Arg}\, z_2$, then $\mathrm{Re}\,(z_1\overline{z}_2) = r_1 r_2 = |z_1||z_2|$. Now, $|z_1 + z_2| = |z_1| + |z_2| \iff z_1\overline{z}_1 + z_2\overline{z}_2 + z_1\overline{z}_2 + z_2\overline{z}_1 = |z_1|^2 + |z_2|^2 + 2|z_1|z_2| \iff z_1\overline{z}_2 + z_2\overline{z}_1 = 2|z_1||z_2| \iff \mathrm{Re}(z_1\overline{z}_2 + z_2\overline{z}_1) = \mathrm{Re}(z_1\overline{z}_2) + \mathrm{Re}(z_2\overline{z}_1) = 2|z_1||z_2| \iff \mathrm{Re}(z_1\overline{z}_2) = |z_1||z_2|$ and $\mathrm{Re}(\overline{z}_1 z_2) = |z_1||z_2| \iff \mathrm{Arg}\,(z_1) = \mathrm{Arg}\,(z_2)$.
3.15. If a is a positive real number, then $\sqrt{a}$ denotes the positive square root of a. However, if w is a complex number, what is the meaning of $\sqrt{w}$? Let us try to find a reasonable definition of $\sqrt{w}$. We know that the equation $z^2 = w$ has two solutions, namely $z = \pm\sqrt{|w|}e^{i(\mathrm{Arg}\, w)/2}$. If we want $\sqrt{-1} = i$, then we need to define $\sqrt{w} = \sqrt{|w|}e^{i(\mathrm{Arg}\, w)/2}$. However, with this definition, the expression $\sqrt{w}\sqrt{w} = \sqrt{w^2}$ will not hold in general. In particular, this does not hold for $w = -1$.
3.16. Use $1 - i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$ and $8i - 8\sqrt{3} = 16\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$.
3.17. Use $|w - z|^2 = (w - z)(\overline{w} - \overline{z})$.
3.18. (a). $z^2 = 2i = 2e^{i\pi/2},\ z = \sqrt{2}e^{i\pi/4},\ \sqrt{2}\exp\left[\frac{i}{2}\left(\frac{\pi}{2} + 2\pi\right)\right]$,
(b). $z^2 = 1 - \sqrt{3}i = 2e^{-i\pi/3},\ z = \sqrt{2}e^{-i\pi/6},\ \sqrt{2}e^{i5\pi/6}$,
(c). $z^4 = -16 = 2^4 e^{i\pi},\ z = 2\exp\left[i\left(\frac{\pi+2k\pi}{4}\right)\right],\ k = 0, 1, 2, 3$,
(d). $z^4 = -8 - 8\sqrt{3}i = 16e^{i4\pi/3},\ z = 2\exp\left[\frac{i}{4}\left(\frac{4\pi}{3} + 2k\pi\right)\right],\ k = 0, 1, 2, 3$.
3.19. Multiply $1 + z + z^2 + \cdots + z^{m-1}$ by $1 - z$.
3.20. Suppose all the roots are real. Let $z = x$ be a real root. Then $a + ib = \left(\frac{1+ix}{1-ix}\right)^n$ implies that $|a + ib|^2 = \left|\frac{1+ix}{1-ix}\right|^{2n} = \left(\frac{1+x^2}{1+x^2}\right)^n = 1$, and hence $a^2 + b^2 = 1$. Conversely, suppose $a^2 + b^2 = 1$. Let $z = x + iy$ be a root. Then we have $1 = a^2 + b^2 = |a + ib|^2 = \left|\frac{(1-y)+ix}{(1+y)-ix}\right|^{2n} = \left(\frac{(1-y)^2+x^2}{(1+y)^2+x^2}\right)^n$, and hence $(1 + y)^2 + x^2 = (1 - y)^2 + x^2$, which implies that $y = 0$.
3.21. Verify directly.