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An Introduction to Complex Analysis

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Lecture 10
Mappings by Functions I

In this lecture, we shall present a graphical representation of some elementary functions. For this, we will need two complex planes representing, respectively, the domain and the image of the function.

Consider z- and w-planes with the points as usual denoted as $z = x+iy$ and $w = u + iv$. We shall visualize the function $w = f(z)$ as a mapping (transformation) from a subset of the z-plane (domain of f) to the w-plane (range of f).

The mapping

$$w = Az \tag{10.1}$$

is known as *dilation*. Here, A is a nonzero complex constant and $z \neq 0$. We write A and z in exponential form; i.e., $A = ae^{i\alpha},\ z = re^{i\theta}$. Then,

$$w = (ar)e^{i(\alpha+\theta)}. \tag{10.2}$$

From (10.2), it follows that the transformation (10.1) expands or contracts the radius vector representing z by the factor $a = |A|$ and rotates it through an angle $\alpha = \arg A$ about the origin. The image of a given region is therefore geometrically similar to that region. Thus, in particular, a dilation maps a straight line onto a straight line and a circle onto a circle.

The mapping

$$w = z + B \tag{10.3}$$

is known as *translation*; here, B is any complex constant. It is a translation, as can be seen by means of the vector representation of B; i.e., if $w = u + iv,\ z = x + iy$, and $B = b_1 + ib_2$, then the image of any point (x, y) in the z-plane is the point $(u, v) = (x + b_1, y + b_2)$ in the w-plane. Since each point in any given region of the z-plane is mapped into the w-plane in this manner, the image region is geometrically congruent to the original one. Thus, in particular, a translation also maps a straight line onto a straight line and a circle onto a circle.

The general *linear mapping*

$$w = Az + B, \quad A \neq 0, \tag{10.4}$$

is an expansion or contraction and a rotation, followed by a translation.

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Example 10.1. The mapping $w = (1+i)z+2$ transforms the rectangular region in Figure 10.1 into the rectangular region shown in the w-plane. This is clear by writing it as a composition of the transformations

$$Z = (1+i)z \quad \text{and} \quad w = Z+2.$$

Since $1+i = \sqrt{2}\exp(i\pi/4)$, the first of these transformations is an expansion by the factor $\sqrt{2}$ and a rotation through the angle $\pi/4$. The second is a translation two units to the right.

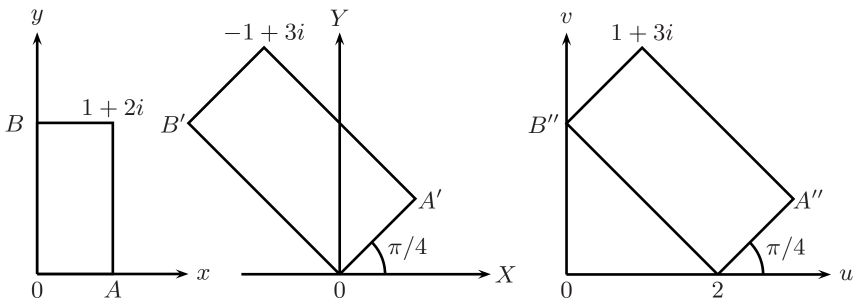

Figure 10.1

The mapping

$$w = z^n, \quad n \in \mathbb{N}, \tag{10.5}$$

in polar coordinates can be written as

$$\rho e^{i\phi} = r^n e^{in\theta}.$$

Thus, it maps the annular region $r \ge 0,\ 0 \le \theta \le \pi/n$, of the z-plane onto the upper half $\rho \ge 0,\ 0 \le \phi \le \pi$, of the w-plane. Clearly, this mapping is one-to-one.

Example 10.2. Let S be the sector $S = \{z : |z| \le 2,\ 0 \le \arg z \le \pi/6\}$. Find the image of S under the mapping $w = f(z) = z^3$. Clearly, we have

$$f(S) = \{w : |w| \le 8, \ \ 0 \le \arg w \le \pi/2\}.$$

Example 10.3. Let S be the vertical strip $S = \{z = x+iy : 2 \le x \le 3\}$. Find the image of S under the mapping $w = f(z) = z^2$. Since $w = x^2 - y^2 + 2ixy$, a point (x, y) of the z-plane maps into $(u, v) = (x^2 - y^2, 2xy)$ in the w-plane. Now, eliminating y from the equations $u = x^2 - y^2$ and $v = 2xy$, we get

$$u = x^2 - \frac{v^2}{4x^2}.$$

Thus, a vertical line in the z-plane; i.e., $x = x_0$ fixed, maps into a leftward-facing parabola with the vertex at $(x_0^2, 0)$ and v-intercepts at $(0, \pm 2x_0^2)$.