defined for all P on $\mathbb{S}$, with only one exception: the north pole N. If P approaches N, then the distance of the corresponding point z in the plane to the origin gets arbitrarily large. This observation shows that the north pole on $\mathbb{S}$ plays the same role as the point at infinity with respect to the complex plane.

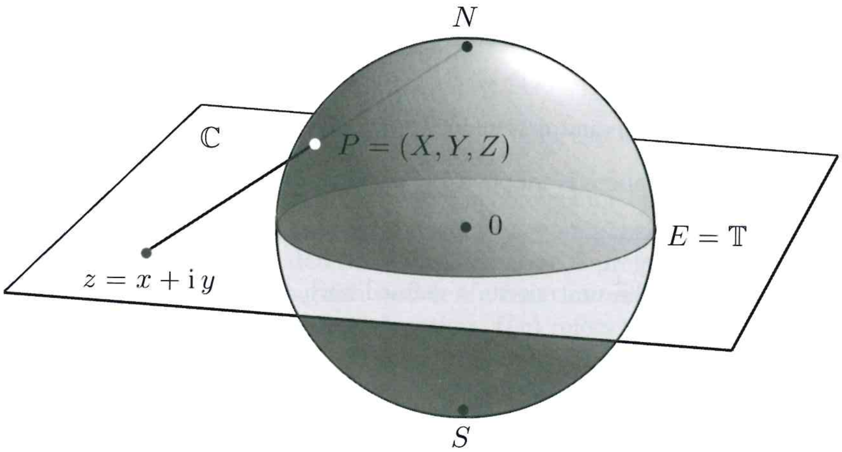

Figure 2.5: Stereographic projection of a sphere onto the complex plane

Extending the stereographic projection to all of $\mathbb{S}$ by assigning the north pole N to the point at infinity results in a bijective correspondence between $\mathbb{S}$ and $\widehat{\mathbb{C}}$. Hence we can label the points on $\mathbb{S}$ with the corresponding complex numbers of $\widehat{\mathbb{C}}$. In what follows we shall therefore identify the sphere $\mathbb{S}$ and the extended complex plane $\widehat{\mathbb{C}}$ and call it the *Riemann sphere*. The *spherical distance* $d(z_1, z_2)$ of two points in $\widehat{\mathbb{C}}$ is the Euclidean length of the straight segment connecting the corresponding points on the sphere $\mathbb{S}$. If $z_1, z_2 \in \mathbb{C}$ then

$$d(z_1, z_2) = \frac{2\,|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, \qquad d(z_1, \infty) = \frac{2}{\sqrt{1+|z_1|^2}}.$$

Arithmetic on the Sphere. Stereographic projection allows us to transplant the arithmetic operations from $\mathbb{C}$ to $\widehat{\mathbb{C}}$. Additionally we postulate that

$$\begin{aligned} z/\infty &:= 0 && \text{for} \quad z \in \mathbb{C} \\ z/0 &:= \infty && \text{for} \quad z \in \mathbb{C} \setminus \{0\} \\ z \pm \infty = \infty \pm z &:= \infty && \text{for} \quad z \in \mathbb{C} \\ z \cdot \infty = \infty \cdot z &:= \infty && \text{for} \quad z \in \widehat{\mathbb{C}} \setminus \{0\}. \end{aligned} \tag{2.15}$$

Note that we do not define $\infty \pm \infty$, ∞/∞, $0/0$ and $0 \cdot \infty$. After extending modulus $|z|$ and phase $\psi(z)$ to all points of the Riemann sphere by setting

$$|\infty| := \infty, \quad \psi(0) := 0, \quad \psi(\infty) := \infty,$$