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The described pilot experiment focuses on developing students' visuospatial abilities by implementing different models, including tangible and virtual 3D geometric models, in teaching spatial geometry to secondary school students (15 – 20 years old). The case study is presented by thirty-one secondary school students solving the given problems concerning the planar cut sections of the models of a cube. In our experiment, we mainly investigate whether students need support tools in the form of tangible or virtual 3D geometry models to solve spatial geometry problems and, if so, which type of model is most useful for them to be able to solve the problems correctly. We explore how the students work with the appropriate models. The data analysis and presentation of the results obtained are based on Maier's framework. The results of the pilot experiment serve as a basis for our further work aimed at developing the concept of teaching geometry using different kinds of models.

Keywords: Teaching geometry, spatial geometry, visuospatial abilities, implementation, cube models.

Introduction

This paper describes a pilot experiment exploring the possibilities of implementing 3D printed models and other models into mainstream geometry teaching at secondary schools. In doing so, we seek to identify approaches that can increase students' knowledge and skills in geometry. In this way, we are responding to problems described in numerous studies. According to Tambychik et al. (2010), most students need help with visuospatial abilities, spatial orientation, and understanding the relative positions among objects. According to Kivkovich (2015), this is due to a lack of understanding of geometry concepts, mathematical language in geometry, and a poor understanding of previous subject matter. Gloria (2015) describes similar problems students have with geometry. According to her study, students exhibit a weak grasp of geometric concepts, insufficient skills in geometry and mathematics, and low levels of motivation to learn geometry.

Theoretical framework

In today's highly flexible society, which disposes of a great variety of IT technologies, optimizing tools for teaching mathematics and geometry is highly desirable. We face the problem of revising the methods and forms of teaching and finding possibilities and ways to enrich education with modern approaches and technologies. We aim to bring geometry teaching into the 21st century while maintaining the appropriate tools of the past and improving the quality of teaching and student outcomes. The importance of teaching geometry and its relevance to developing spatial visualisation and problem-solving skills have been mentioned by several experts for decades (Čech, 1940-1941; Dehaene et al., 2006). Other authors have reported positive effects of teaching geometry on the growth of mathematical and other cognitive abilities, including IQ (Clements & Sarama, 2007).

Tatsuoka et al. (2004) describe how geometric knowledge is closely related to mathematical reasoning. She believes acquiring geometry skills may be necessary for higher-order mathematical thinking.

Even though experts have a consensus on the importance of geometric knowledge, the results of pupils and students in this area could be more satisfactory. It would seem that the decline in this knowledge and abilities has only occurred in recent years, but several studies indicate otherwise. Some researchers have pointed to low levels of geometric knowledge and poor understanding of concepts (e.g., Unal et al., 2009). These problems are related to more than school geometry: The importance of geometry also lies in its importance in developing students' complex mathematical skills. Visuospatial ability, an essential aspect of human cognition, is linked to geometry education (Maier, 1994). Many works deal with visuospatial ability, and its essential components are also mentioned. For example, Gardner (2011) points out that the visuospatial ability components such as mental manipulation, rotation, bending, or flipping the depicted object are some of the critical aspects of intelligence. Linn & Petersen (1985) categorised visuospatial ability using three modules: mental rotation (the ability to quickly and precisely rotate 2D or 3D objects, imagining the attributes of the resulting object subsequent to its rotation around an axis by a specified number of angular degrees), spatial perception (the ability to identify the spatial relationships of an object concerning the orientation of one's own body), and spatial visualisation (the ability to process complicated spatial information related to an object, such as how its parts are arranged). While Maier (1994) uses the division of visuospatial ability into five components, see Table 1.

Table 1: Aspects of visuospatial ability according to Maier (1994)

Aspects of visuospatial ability		**Description**
A1	*spatial perception*	solvers are demanded to designate spatial relations concerning the orientation of their bodies, despite distracting information
A2	*spatial visualisation*	the ability to visualise the object and its parts in the space
A3	*mental rotation*	the ability to rotate the object in the mind
A4	*spatial relation*	the ability to imagine spatial objects, their parts, and their relationships
A5	*spatial orientation*	the ability to orient oneself in space

The usage of visuospatial representations in solving geometric problems conclusively correlates with problem-solving exercises in general, as described, e.g. by van Garderen & Montague (2003).

With these circumstances in mind, our paper is devoted to the description of a pilot experiment dedicated to the possibility of implementing different types of models, including models created from modelling clay, 3D printed models, and virtual 3D models, in geometry teaching at secondary schools. Their essential role in many scientific disciplines and technological fields is highlighted by, e.g. Mulligan (2015). We are convinced that teaching and teachers play a significant role in this respect. They are ultimately the main organisers of teaching. However, some studies mention the low level of geometry knowledge of primary and secondary school teachers (Unal et al., 2009).

The importance of visualisation in teaching geometry and the related use of models is mentioned in some papers. Risma et al. (2013) also reported promising results using cubic solids in developing visuospatial skills. Huleihil (2017) describes significant improvements in students' understanding of geometry through 3D printed models. A further advantage of using educational tools and activities related to them in teaching geometry is their potential to motivate students. A motivated student becomes an active and engaged element in the learning process. Engagement is defined as behaviours and emotions reflecting the student's involvement in problem-solving (Skinner et al., 2009). The potential for internal motivation increases in such learners, positively affecting their engagement in the learning process (Deci & Ryan, 2000). When students are actively involved in the whole learning process, a deeper understanding of the subject matter, the emergence of more lasting knowledge, and a more effective process are then achieved. Using models, educational tools, and appropriate software as a complex unit contributes to a faster and more thorough understanding of the content by students, so learning becomes more meaningful to them (Prensky, 2010).

Despite the benefits of using different models, many secondary mathematics teachers rely only on textbooks and worksheets (Sriraman, 2005). While such instruction may provide an understanding of basic concepts, it provides little to increase student interest or deeper engagement in these areas (Coxbill et al., 2013). Jaakkola & Nurmi (2008) made pairwise comparisons among students who used illustrative models and hands-on learning activities with computer simulations with those who experienced traditional lecture-type instruction. They found that students actively engaged in learning using real-world models and simulations achieved significantly higher learning outcomes. Hobson et al. (2010) demonstrated that students gained more profound insight into the issues discussed by linking computer-based virtual models to physical models. 3D printing can also effectively link the two areas - tangible models and computer simulations, along with active student involvement in the learning process. Proper use of its aspects in teaching geometry leads to a better understanding of geometry, developing students' visuospatial skills and increasing their mathematical and abstract thinking (Dilling & Witzke, 2020).

After implementing the developed tools, we presume the teaching will potentially increase students' geometric thinking. Moreover, it could naturally lead to better development of students' visuospatial abilities, correct understanding of concepts, and a deeper understanding of geometry. In this way, we would also like to contribute to increasing students' engagement in learning. We draw on constructivist approaches and utilise activating ones (Zormanová, 2012).

Methods section

Thirty-one 17-18-year-old students in the secondary school in Liberec started studying spatial geometry, respectively solving the positional problems directed to constructing planar cut sections of angular solids on January 2023. Their mathematics teacher used the black wire edge model of a cube in connection with her hands to demonstrate the relative positions of straight lines in connection to this model. Further, she sketched the most straightforward problems for constructing planar cut sections of the cube model in the free parallel projection on the blackboard. The students were asked to redraw the solutions into their worksheets. Unfortunately, most students in the classroom could not imagine the required spatial situations and understand the solved problems. The mentioned facts confirm the observations described by Sriraman (2005).

We learned about the situation in the third year of this secondary school in Liberec through coincidences and circumstances. We offered the teacher to help her with teaching spatial geometry. We attended the class in two for one month. There are three mathematics lessons per week in the third year, so we attended twelve teaching hours in the classroom. One of us was teaching while the other was observing the students' reactions to our questions, understanding the issues discussed, working with the models, recording solutions to the problems in their worksheets, taking photo documentation and audio recordings in the lessons, etc.

We selected the problems assigned to the students in the context of the school curriculum objectives and based on discussion with the mathematics teacher in the class. The teacher told us only the topic to be covered with the students. Because of knowing based on the information we had gathered, the students had not understood the initial tasks from the teacher's explanation so far, so we started from the beginning. We used the experience of teaching future mathematics teachers at the university and first included primary tasks on which solutions to the more complex problems are based. By primary tasks, we refer to exercises that reinforce the fundamental principles of spatial geometry, such as the incidence of points, straight lines, and planes, their relationships, and their relative positions in two or three-dimensional space. These tasks aim to understand the problems of constructing planar cut sections of models of angular solids. Before starting the individual spatial geometry lessons, we placed wooden, 3D printed plastic, paper models of a cube, magnetic "edge" cube model made from Magformers kit, and modelling clay on each desk (see Figures 1 – 4). Students received plastic modelling pads, bamboo toothpicks, skewers, and knives. In most cases, students worked in pairs at their desks and were asked to create a cube model using the modelling clay.

Results

In the beginning, the students were able to list the relative positions of the basic geometric objects mentioned above with a small amount of help from the teacher, then followed the first task to sketch into pre-drawn cube models in the free parallel projection three pairs of straight lines that coincide with the edges of a cube model and are parallel to each other, intersecting, and inclined. The students suggested and drew three pairs of straight lines with no mistakes. Considering the aspects of spatial ability according to Maier (1994) mentioned by Bímová et al. (2022), the students used aspects **A1**, **A2**, and **A4** while solving the task. None felt the need to use any of the provided models to solve the task.

The situation changed when solving the second task. The students were asked to determine the relative positions of pairs of straight lines given by two points each. Compared with the first task, the given straight lines do not coincide with the edges of the cube model. The straight lines either lay in the faces of the cube model or pass through it. In this case, most students in the classroom could not distinguish the relative positions of the given pairs of straight lines only in their minds. The teacher encouraged them to use one of the models they had available. Approximately one-quarter of the students resolved the task by sketching images of the given pairs of straight lines into pre-drawn cube models in the free parallel projection. Those students could resolve the task by using aspects **A1**, **A2**, **A4**, and **A3,** according to Maier (1994). The second quarter of the students used the magnetic "edge" cube model made from the Magformers kit. Working with this kind of model, the students could simulate images of the given straight lines passing through the cube model (see Figure 1, part a); they had to make a picture of the simulated situation in their minds because when using both their hands to hold bamboo skewers it is impossible to sketch the images of the particular pairs of the given straight lines into their worksheets at the exact moment. Half the students applied a cube model from the modelling clay. If the students modelled precisely, they received the relatively exact models of the spatial situations and could find the relative positions of pairs of given straight lines. Cube models made from modelling clay hold together, including added bamboo skewers to represent the models of the pairs of straight lines (see Figure 1, parts b and c); students were able to rotate them and look at them from different points of view, thus getting involved in the aspect **A5** in the problem-solving process in addition to the **A1**, **A2**, and **A4** aspects according to Maier (1994). This group of students could redraw the spatial situations into pre-drawn cube models in the free parallel projection in their worksheets according to the cube models laid out in front of them. Only the individuals used a wooden or 3D printed plastic or paper cube model (see Figure 1, part d). This category of model necessitated considerable involvement in hands-on activities.

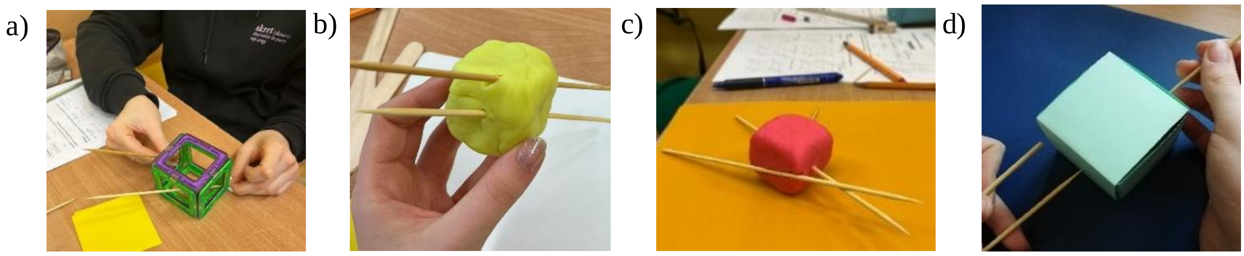

Figure 1: Using different kinds of models for simulating relative positions of pairs of straight lines

In the next problem, students had to determine the relative positions of a straight line and a plane. Here they preferably used the magnetic "edge" model of the cube made from the Magformers kit or the cube models made from modelling clay. The first imperfections in the representation of planes on models of the cube drawn in the free parallel projection began to appear. Students did not correctly create all sides of the planar cut section of the cube in their drawings, which in the given task coincided with either the edges of the cube models or the face diagonals. In these cases, they were helped by a sheet of paper inserted into the internal part of the magnetic "edge" model of the cube made from the Magformers kit or by actual cuts of the cube models made from modelling clay performed with bamboo knives. See Figure 2. Students started realising what the planar cut section of the cube model meant.

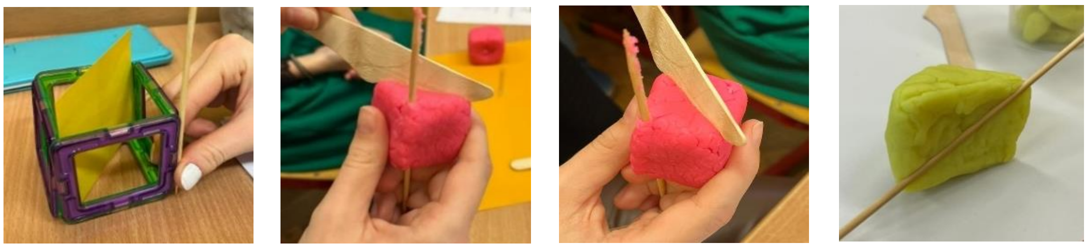

Figure 2: Using two kinds of models for simulating relative positions of a straight line and a plane

This was followed by a task in which students had to think out and then sketch into the pre-drawn models of the cube in the free parallel projection on the blackboard and on their worksheets all the polygons that could be formed by planar cutting the cube model. Several students started thinking and sketched possible polygonal cuts into their worksheets. Some other students did not waste any time and took advantage of their cube model made of modelling clay, used a bamboo knife, and tried to create different kinds of polygonal cuts (see Figure 3, part a). Using models made from modelling clay was sufficient to create the initial idea of polygonal sections of the cube. However, these models no longer successfully introduced the basic principles of constructing planar cut sections. The models were relatively soft and deformed when cut with bamboo knives. Some definitions and theorems, valid in spatial geometry, were thus violated due to model deformations. At this stage, the 3D printed cut cubes' models helped show the students the possible polygonal sections, especially the valid principles in constructing planar cut sections of the cube (see Figure 3, parts b and c).

Figure 3: Planar cut sections on the model made from modelling clay and on 3D printed models

After introducing the basic rules used in the constructions of planar cut sections, after practising more straightforward problems in which it is possible to construct the sides of the polygonal sections either only by drawing a straight line passing through two points lying in one face of a cube model or by drawing parallels lying in parallel faces of a cube model in the following mathematics lessons, we then switched to plot the planar cut sections using axial affinity. When solving problems of this type, students of the whole class except one student only passively received information about the individual steps of the constructed planar cut sections. They waited for the objects to be displayed in a sequence on cube models drawn on the blackboard or to appear in dynamic applets projected on the projection screen. The students gradually redrew the individual constructed objects into the pre-drawn cube models in the worksheets. One student first determined the imaginary intersection of two inclined straight lines during the construction but then realised his mistake and corrected himself correctly by saying, “Oh, yes, this straight line lies on the back face”.

Initially, students did not use the models they had on their desks. Only one student modelled all straight lines using bamboo skewers, leading to correctly constructing a planar cut section on the cube model made from the modelling clay (see Figure 4, part a). He modelled alone, rotating the model from different points of view. He did not plot the process solution into his worksheet. However, he correctly advised his classmate on constructing the particular objects on the pre-drawn cube model in the free parallel projection. He was the only one able to react correctly to all the questions asked by the teacher. After starting to use models, the students reacted more appropriately and began to be more active. The constructed planar cut sections of cube models were drawn on the blackboard and in the dynamic applets (see Figure 4, part b). Some students redrew them correctly, including distinguishing the visible and invisible sides of the polygonal sections. Others, although they could see the differentiation of the visible and invisible sides of polygonal sections on the blackboard or in dynamic applets, did not correct the corresponding sides of the planar cut sections using complete and dashed lines. Next, we switched the constructions displayed in the GeoGebra 3D window into the version for anaglyph glasses (see Figure 4, part c). Students could look at the constructions through the lent anaglyph glasses. Some students reacted spontaneously when switching the construction displayed in the dynamic applet to the anaglyph glasses version, expressing their sudden understanding. Finally, we presented to the students the overlap leading to the generation of a virtual model as a basis for 3D printing of the physical model from the constructed virtual model displayed in the 3D window of GeoGebra. We 3D printed the physical model and lent it to the students for viewing (see Figure 4, part d). This time, another group of students also understood the principles of constructing a planar cut section of a cube model constructed using axial affinity.

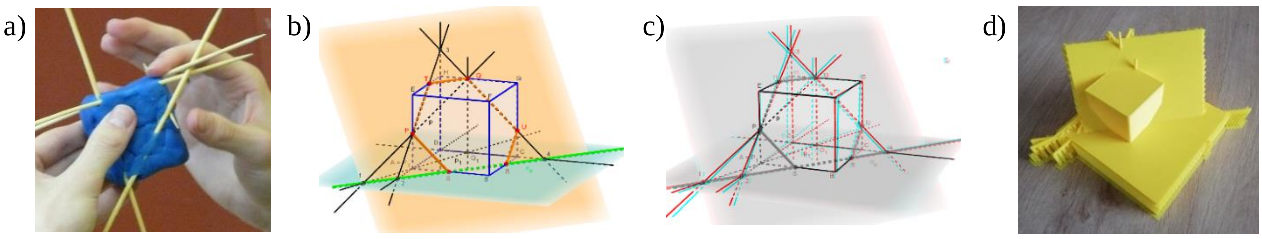

Figure 4: Planar cut section of the cube displayed on different kinds of models

Conclusion

In this paper, we described the initial parts of our pilot experiment: implementing 3D printed models and other models in teaching spatial geometry. The pilot experiment confirmed the results reported by Risma et al. (2013) and Huleihil (2017), stating that using not only printed models by 3D printing significantly contributes to improving students' understanding of geometry. Various students in the classroom were helped to understand the relative positions of objects. Subsequently, they were helped with solving problems focused on constructing planar cut sections of the cube models using different models. In the following lessons, we will continue constructing planar cut sections of cube models and related examples by students in the virtual environment of GeoGebra software, with students being asked to model such planar cut cube models that, once 3D printed, could help them most effectively. From such activities, it is easy to see which models help students best.

We will report on the results of the ongoing phases of the pilot experiment in our following papers. The mathematics teacher has already asked us to integrate the same activation methods and supporting models into teaching spatial geometry in the seventh grade of the eight-year secondary school in Liberec.

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