Pädagogische Hochschule Schwäbisch Gmünd

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Possibilities to connect arithmetic and geometry – chance to realize child-centered math-lessons

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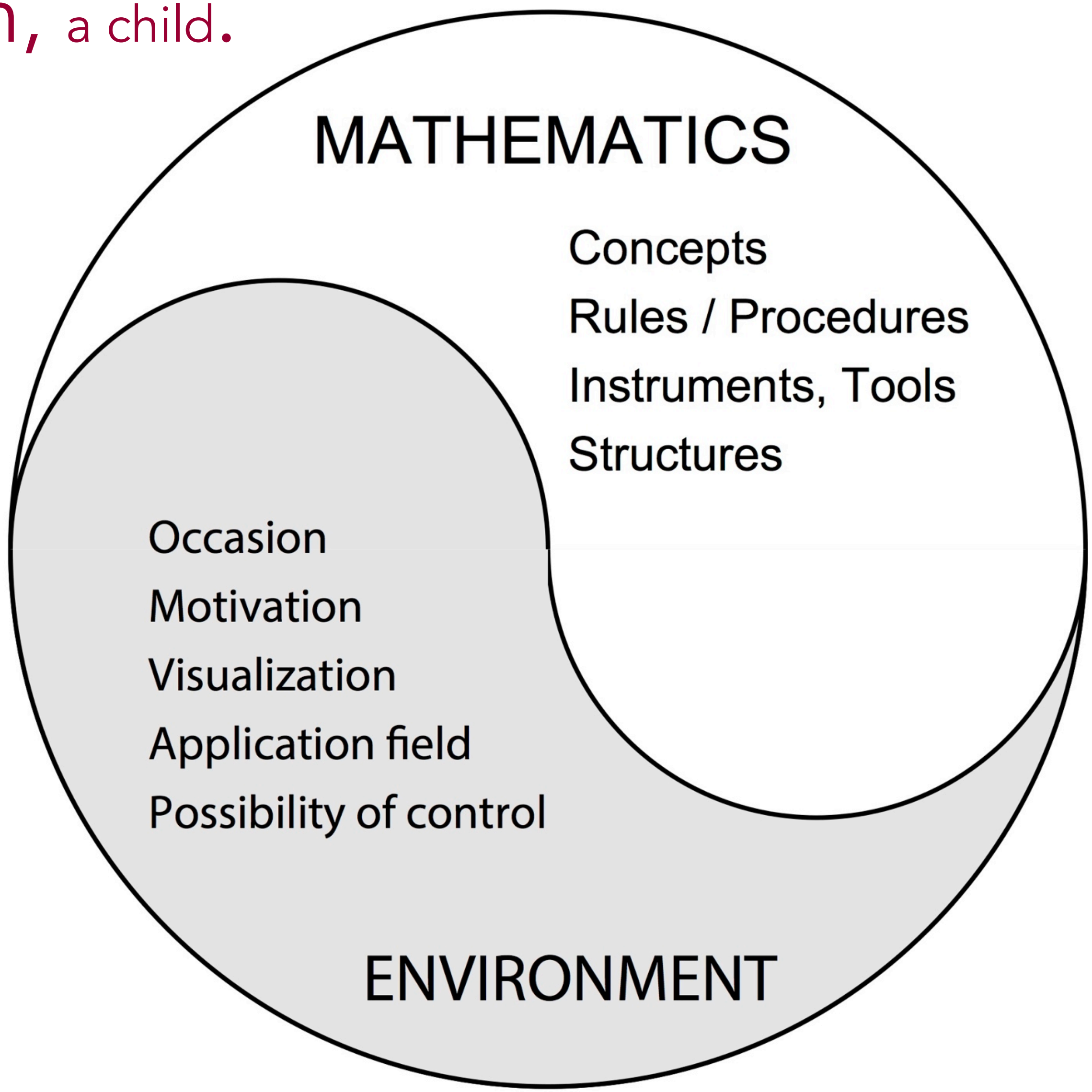
The world, mathematics and between them, a child.

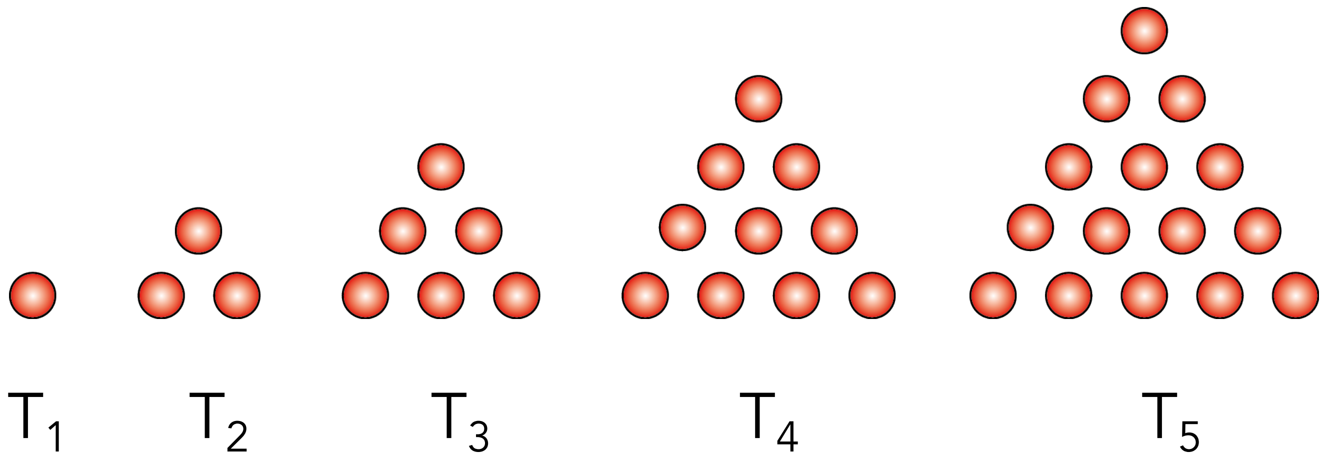


Historical notes

- Mathematics has a unique epistemological character.
- Since ancient Greek times, proofing has been a crucial activity in mathematics. Proofing was at the center of the creation of new knowledge.
- The possibility of geometrical visualization has always played an important role.
- Especially the study of number patterns has played a central role since Pythagoras of Samos (ca. 600-500 B.C.)

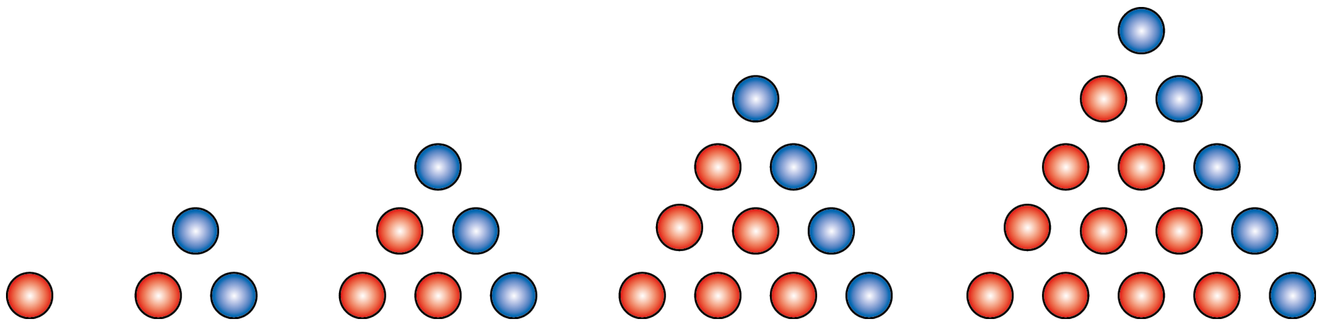
Historical notes

- The New Pythagorean Nikomachos of Gerasa (ca. 60-120 A.D.) intensively investigated triangular, quadrilateral and pentagon figures.



Historical notes

- Clever laying of point patterns provided non-trivial insights even then.



Historical Notes

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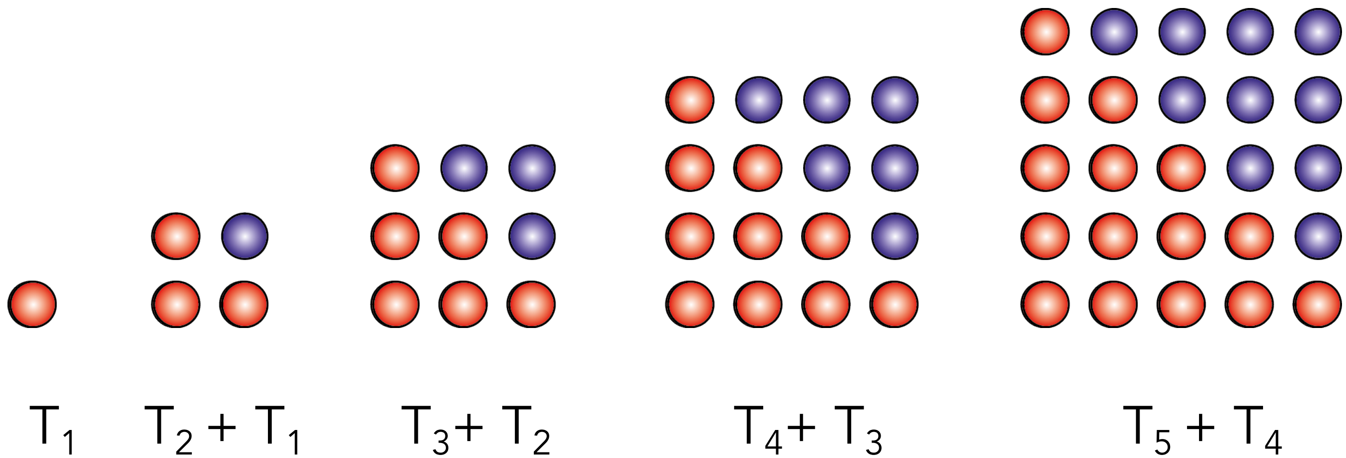
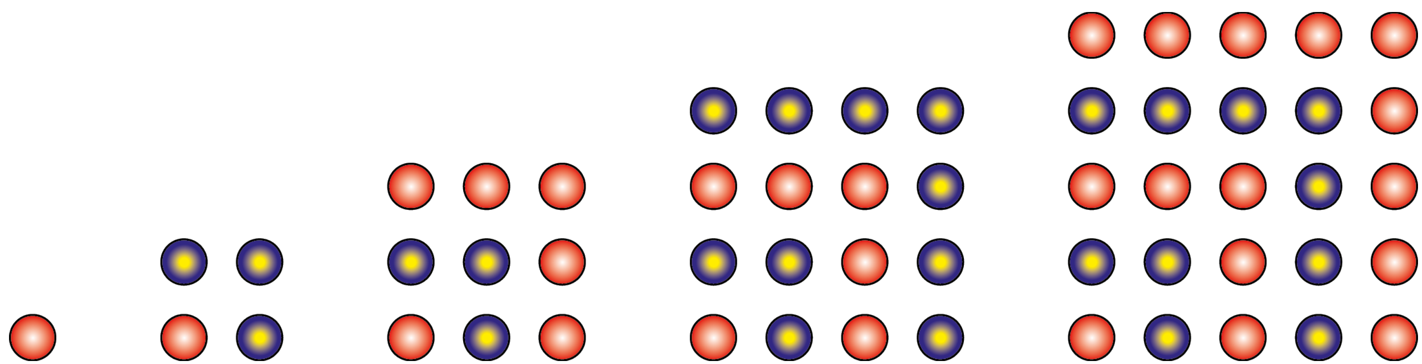


Historical Notes

- Angle hooks (so-called "gnomons") were often used for visualization:



1   1 + 3   1 + 3 + 5   1 + 3 + 5 + 7

Which equation fits?

Historical Notes

- Even later great mathematicians often worked with the technique of figured numbers or other visualizations.
- e.g., Carl Friedrich Gauss (1777-1855):
- 1 + 2 + 3 + … + 99 + 100 =

Porträt made by Gottlieb Biermann (1887)
(c) Picture: A. Wittmann, Gauß-Gesellschaft Göttingen e.V.

Historical Notes: C. F. Gauss …

$1 + 2 + 3 + \ldots + 98 + 99 + 100 = 5050$

How so fast?

$$
\begin{array}{rrrrrrr}
1 + & 2 + & 3 + & \ldots + & 98 + & 99 + & 100 \\
100 + & 99 + & 98 + & \ldots + & 3 + & 2 + & 1 \\
\hline
101 + & 101 + & 101 + & \ldots + & 101 + & 101 + & 101
\end{array}
$$

$(100 \cdot 101) : 2 = 5050$

today

Learning mathematics is …

- a process of active construction of meaning,
- a social process,
- a process that requires a focused, guided confrontation.

Today: active construction of meaning

$$25 \cdot 36$$

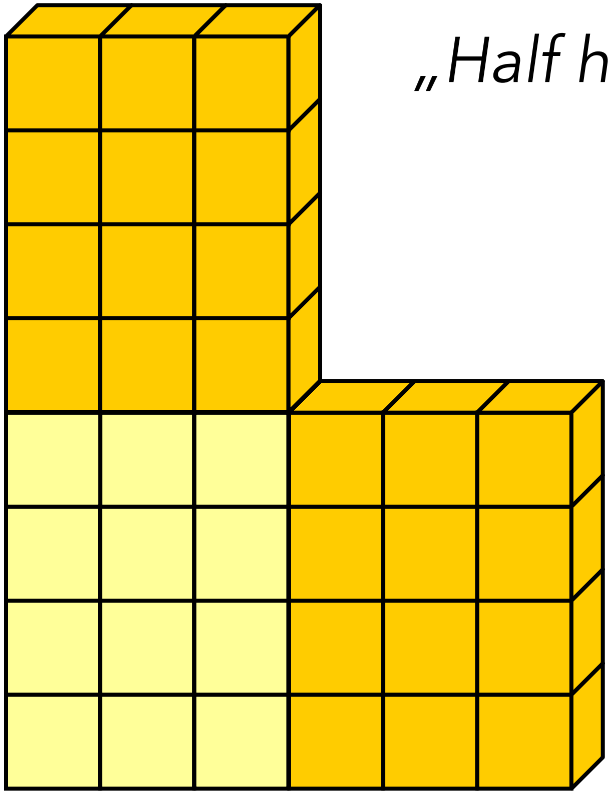
$$100 \cdot 9 = 900$$

- „Allows“ the teacher this „trick“?

or

- can the students see why this is correct?

Today: active construction of meaning



„Half height means double width."

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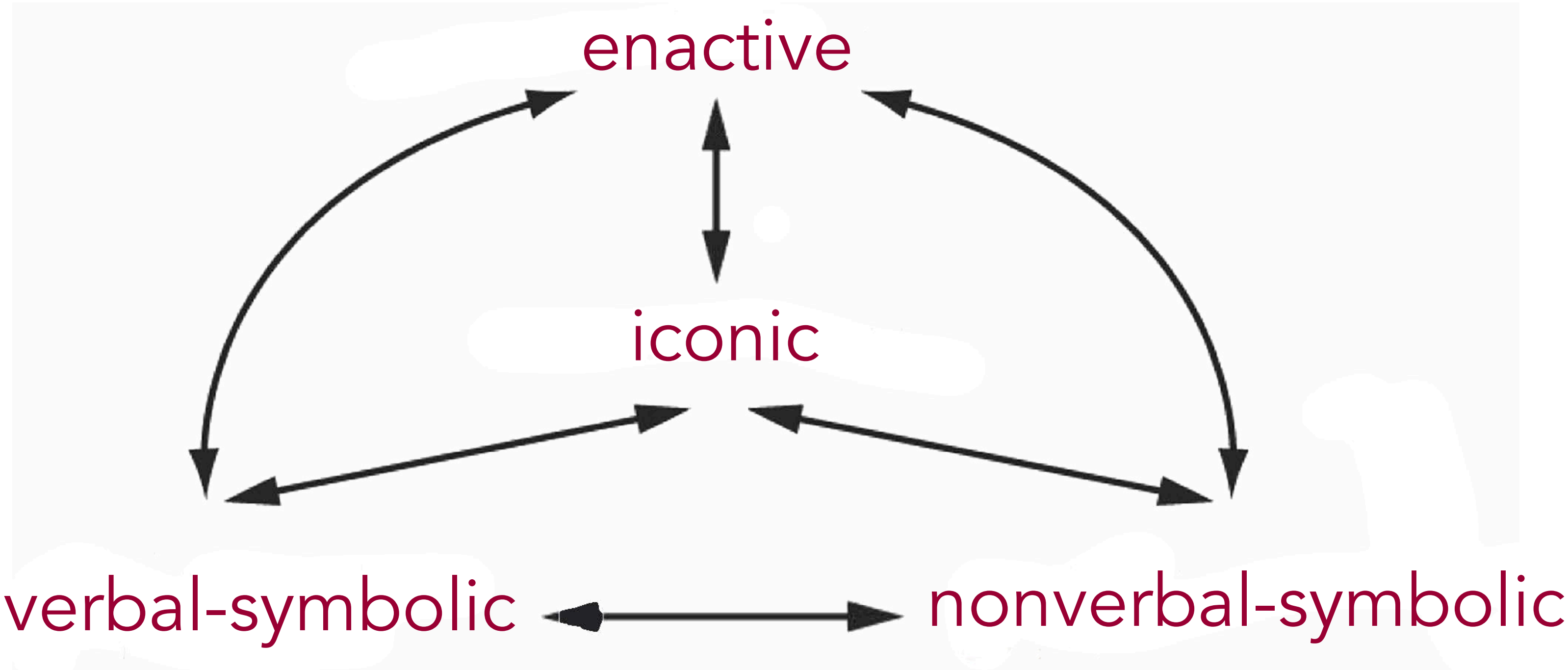
Today: active construction of meaning

- $25 \cdot 36 = ?$
- How you find the solution? Describe!
- Why your way is correct?
- Why is $\frac{1}{2} : \frac{1}{4} = 2$?
- Why is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$?
- How you figure out 25% of a way?
- How you figure out 25% of 360 CZK?

J. Bruner



Important:

intermodal transfer

verbal-symbolic / non-verbal

enactive level is not just a short state!

Facts

Mental pictures support thinking

The connection of the ability of mental visual operation and dyscalculia is well shown. (Lorenz 1992)

About 80% of children with dyscalculia show a lack in the ability of visualization.

Facts

INTERACTIONS BETWEEN NUMBER AND SPACE IN PARIETAL CORTEX

Edward M. Hubbard, Manuela Piazza, Philippe Pinel and Stanislas Dehaene

Abstract | Since the time of Pythagoras, numerical and spatial representations have been inextricably linked. We suggest that the relationship between the two is deeply rooted in the brain's organization for these capacities. Many behavioural and patient studies have shown that numerical–spatial interactions run far deeper than simply cultural constructions, and, instead, influence behaviour at several levels. By combining two previously independent lines of research, neuroimaging studies of numerical cognition in humans, and physiological studies of spatial cognition in monkeys, we propose that these numerical–spatial interactions arise from common parietal circuits for attention to external space and internal representations of numbers.

Hubbard, E. M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. Nature Reviews Neuroscience, 6(6), 435- 448. doi:10.1038/nrn1684

Facts

sory, world-centred representations of space[144,145]. This implies that the same computational transformations that support spatial updating are crucial for arithmetic operations that create shifts in the locus of activation along an internal number line (FIGS 4 and 5). Indeed, the problem of computing a world-centred spatial representation by combining two separate population codes for eye and retinal location is formally identical to that of computing an approximate addition or subtraction by combining two population codes for numerosity[144,145]. Therefore, the parietal mechanisms that are thought to support spatial transformations might also be ideally suited to supporting arithmetic transformations.

Future studies can test this prediction by compar-

(Hubbard, Piazza, Pinel, & Dehaene 2005, p. 445)

Facts

In mathematics lessons we always have to visualize

- objects
- relations
- processes

Consequencies

Enable the students the perception of:

- objects
- relations
- operations

then challenge the students to imagine:

- objects
- relations
- operations

A modern way of teaching mathematics …

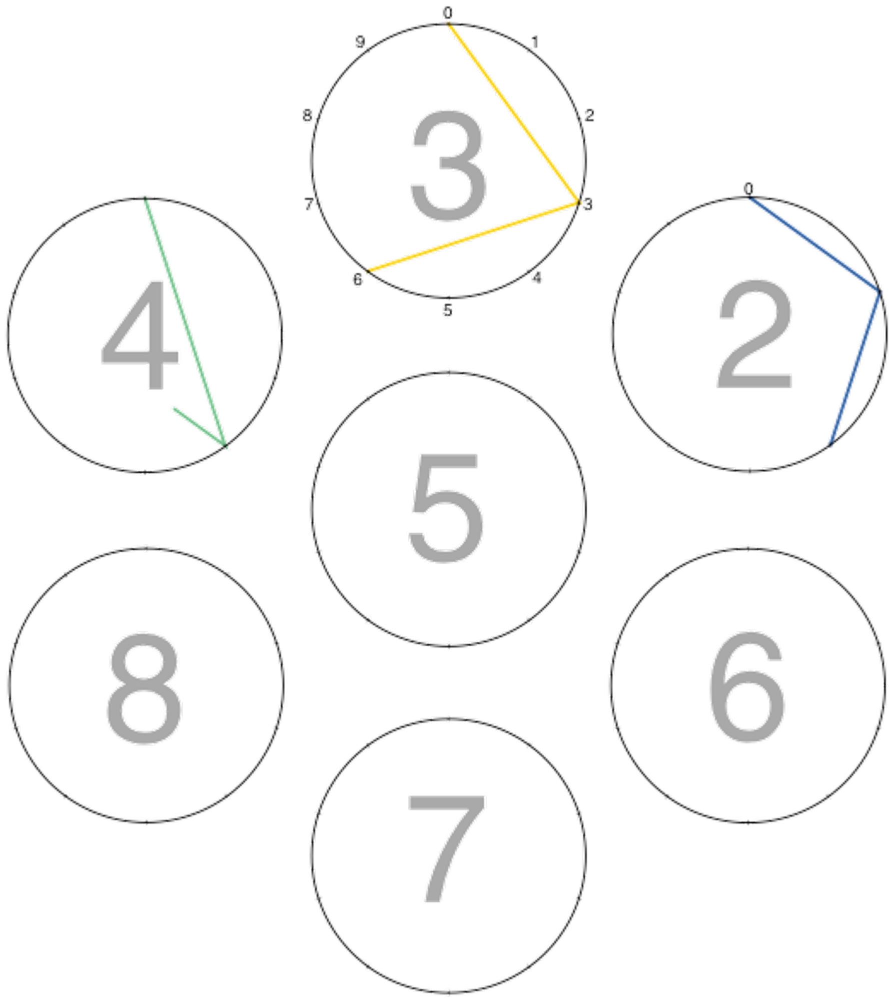
connects arithmetic and geometry:

- Arithmetic facts are illustrated with geometric activities / pictures (e.g. the sum of two even numbers is even)

- Geometrical facts and activities are challenges to arithmetic activities.

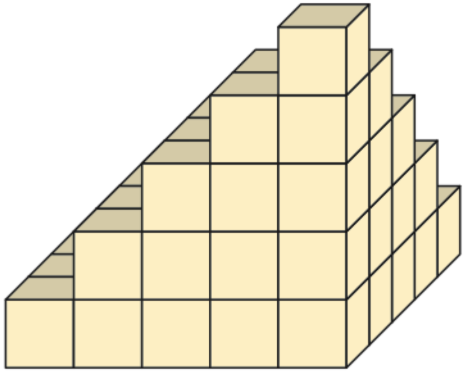
Geometrical as an occasion for arithmetic

Example: Circles of ten



Geometrical as an occasion for arithmetic

Where is arithmetic?



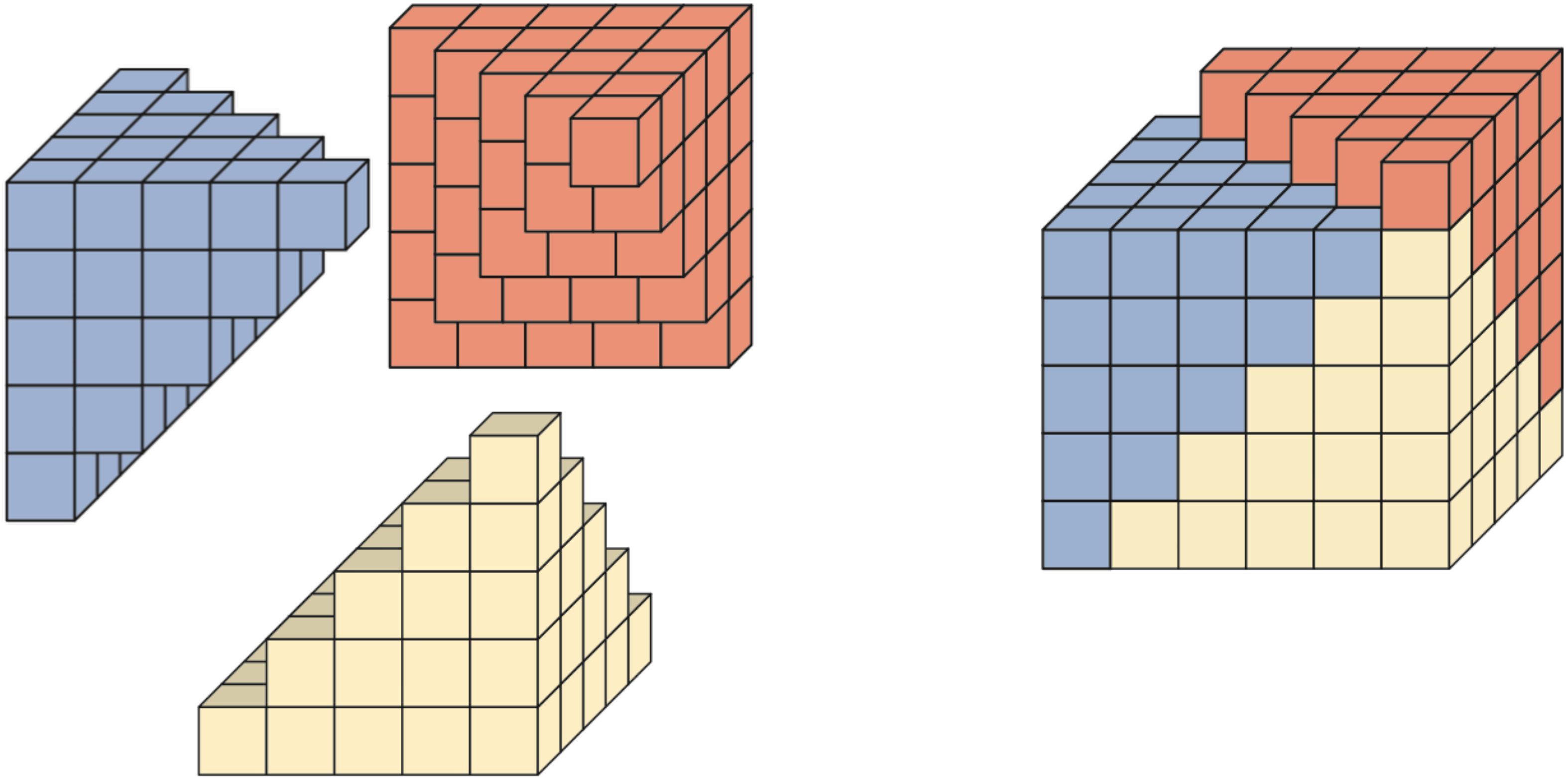
$1^2 + 2^2 + 3^2 + 4^2 + 5^2$

a sum of squares

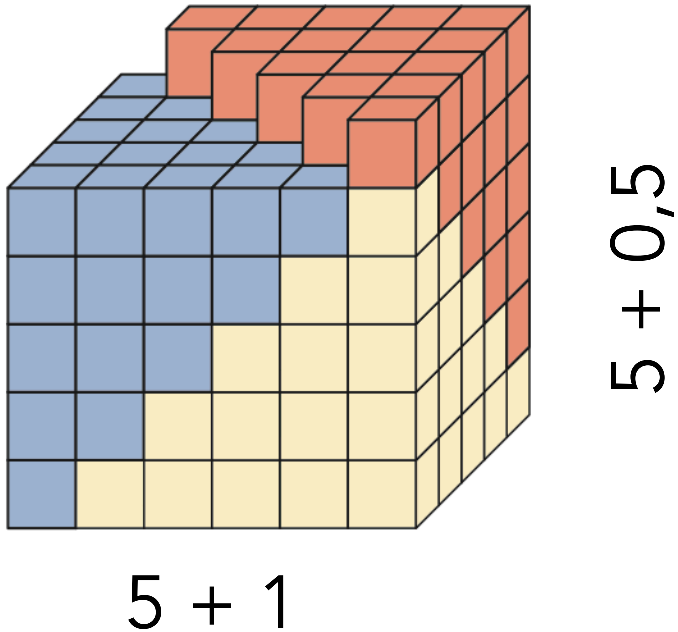
Where is arithmetic?

Imagine a spatial movement:

What is the result? A cube?



That’s arithmetic



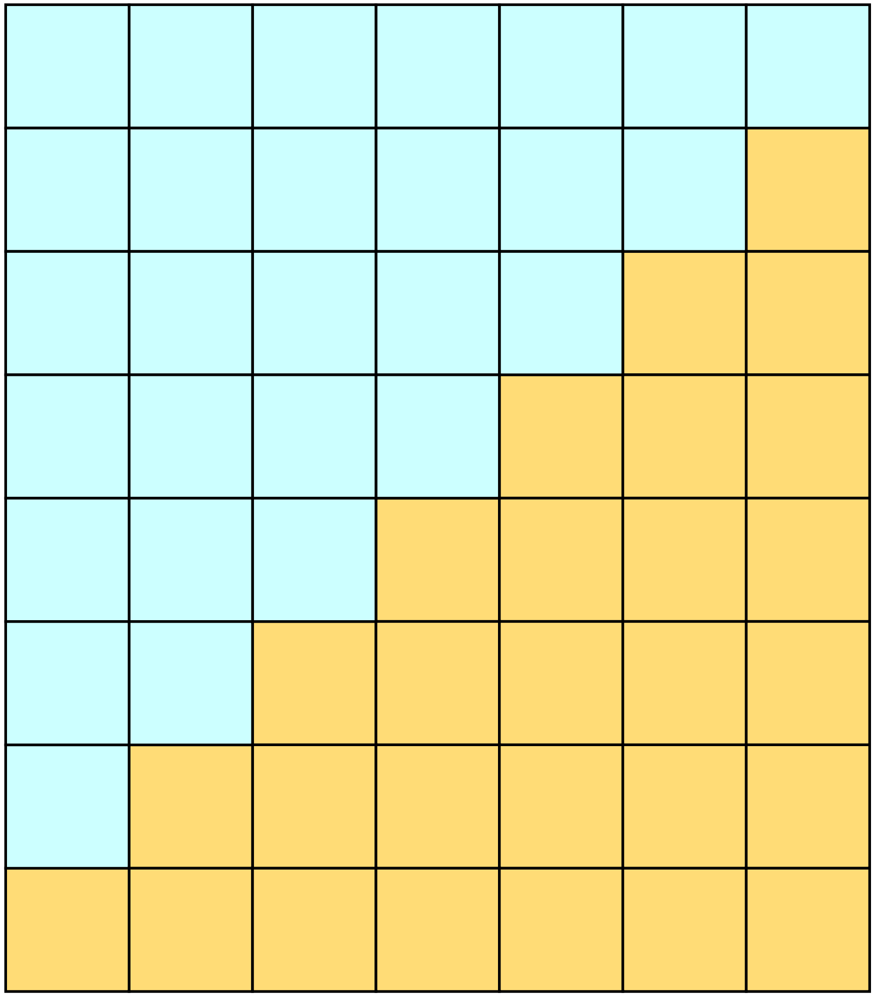
$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = [5 \cdot (5 + 0{,}5) \cdot (5 + 1)] : 3$$

You see it and you will remeber it.

What means „it“? – The body or the formula?

Solving tasks – finding patterns

$1 + 2 + 3 + 4 + 5 + 6 + 7 = 7 \bullet (7 + 1) : 2$



Solving tasks – finding patterns

$1 + 2 + 3 + 4 + 5 + 6 + 7 = 7 \bullet (7 + 1) :$

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = (7 \bullet 7{,}5 \bullet 8) : 3$

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 =$?

Is there a pattern? If yes, what's the pattern?

What is the solution in the third row?

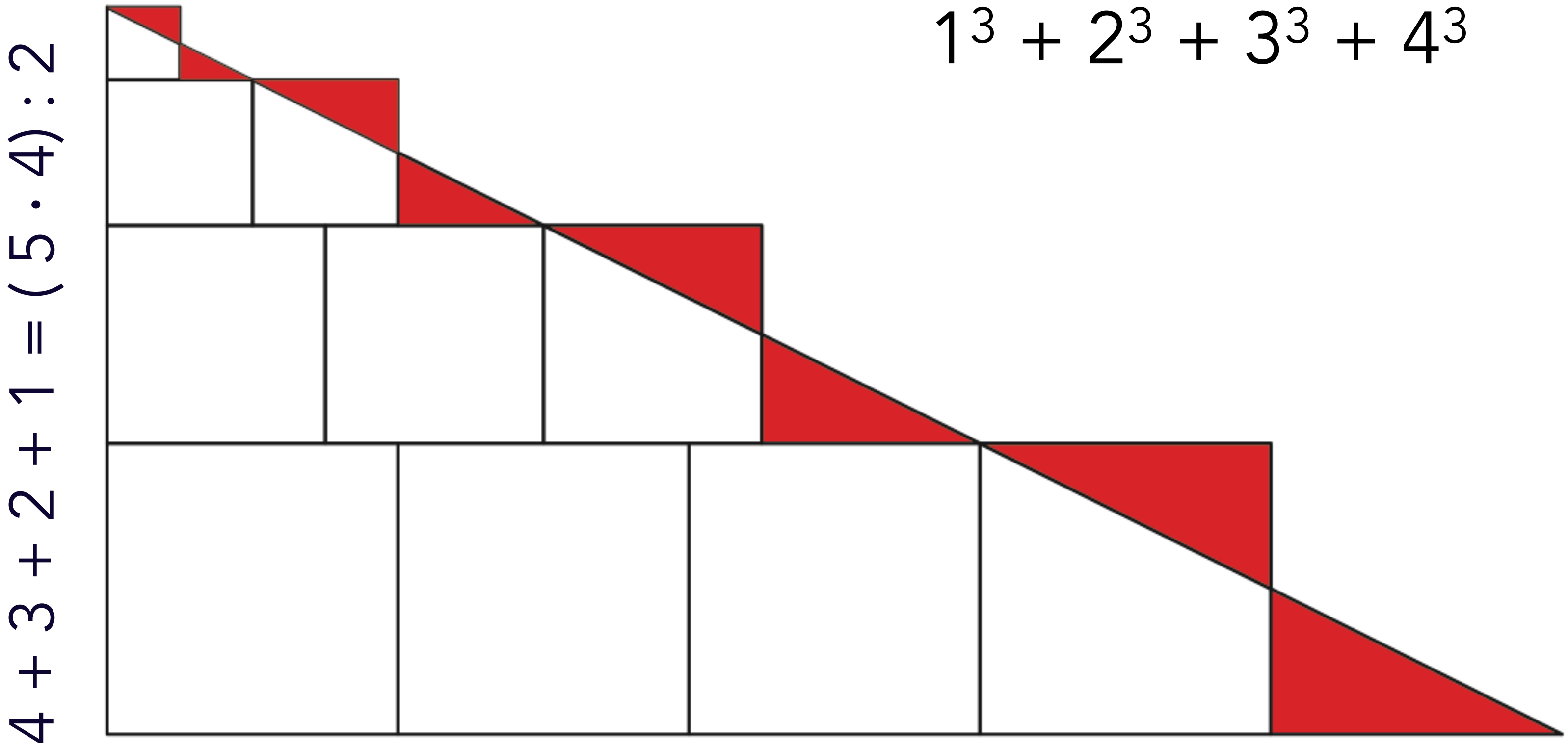
$(7 \bullet 7{,}5 \bullet 8 \bullet 8{,}5) : 4$? $(7^2 \bullet 8^2) : 4$

Can you visualize and see, what it is true?

Let's see ...

Solving tasks – finding patterns

$$1^3 + 2^3 + 3^3 + 4^3$$



$5 \cdot 4$

$$2\,A = [\,5 \cdot 4 \cdot 5 \cdot 4\,] : 2$$

$$A = 5^2 \cdot 4^2 : 4$$

Solving tasks – finding patterns

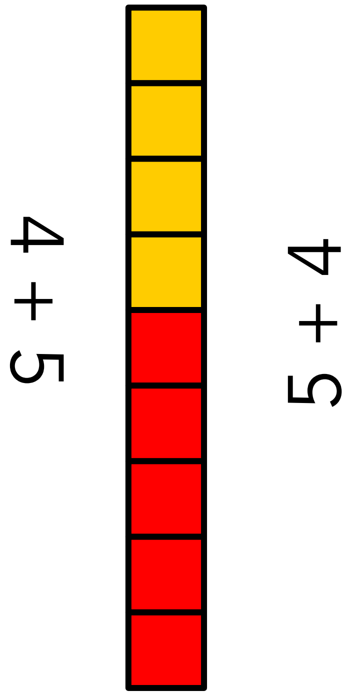
$$1 + 2 + 3 + 4 + 5 + 6 + 7 = 7 \cdot (7 + 1) : 2$$

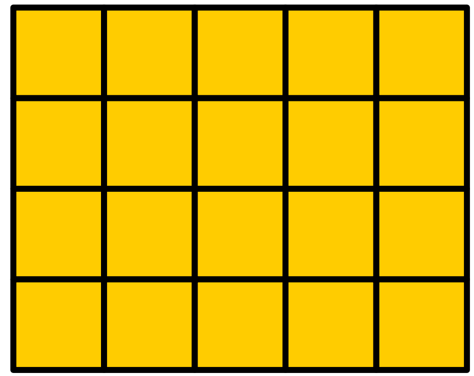
$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = (7 \cdot 7{,}5 \cdot 8) : 3$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = 7^2 \cdot 8^2 : 4$$

visualization of arithmetic facts

Commutativity



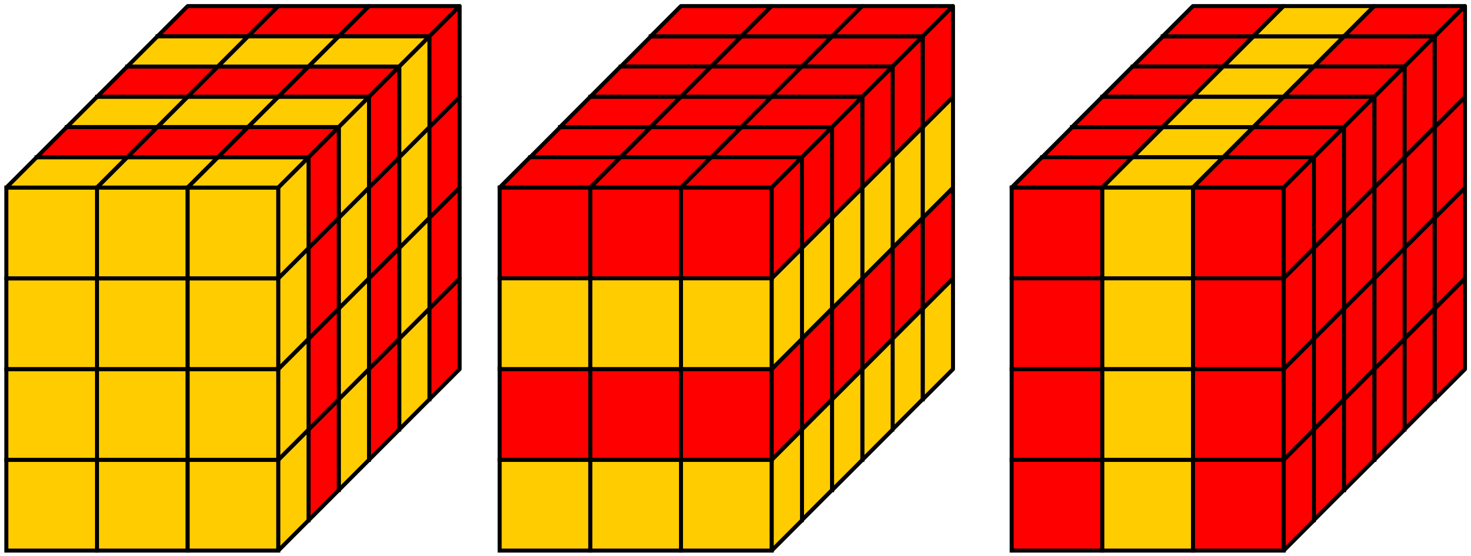


$4 \cdot 5$

$5 \cdot 4$

visualization of arithmetic facts

Which fact?



How to improve your math-lessons?

Students acquire understanding, if they

- have previous perceptions and experiences,
- are familiar with the tools of visualization, the manipulatives,
- are able and accustomed to use geometrical objects as well as productive fantasy as tools of visualization.

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