Language and Mathematics: ambiguity vs precision in today's world

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I would then like to say how I arrived at the contents of this speech, which have changed from the moment I was invited to today, also in view of the events of recent times.

This is how I will try to put the great theme of peace at the center of my speech. But the title has remained what I had thought in past time, because I realized that the reflections on the relationship between mathematics and peace are a natural development of what I had already thought of saying.

I would also like to add that my speech will not deal with, so to speak, technical aspects of mathematics teaching and is not aimed exclusively at primary school, but concerns the whole world of teaching and is indeed an overall reflection on mathematics, not the upper one, but the one accessible to all, and on its relationship with the world in which we live.

Let's get to the point that haunts me and I think it haunts us all. How can you contribute to peace today? For heaven's sake, I do not want to give sermons and I do not even pretend to enter into topics on which I have no qualification and competence to speak, but like everyone I am struck by the latest events, and I ask myself questions.

Certainly the drive to wage war is inherent in human nature, and the opposing interests of different communities of human beings drive them.

But is there nevertheless something - small, limited, but also significant - that the study and practice of mathematics can do to offer a small contribution to building peace? My answer is yes, and I'll try here to say why.

My answer is divided into three points:

1. **Mathematics is universal**
2. **Mathematics is search for precision**
3. **Mathematics focuses on variability and interdependence**

I will speak very briefly about the first point, easy and obvious,
longer about the second, which recalls the title of my speech,
and I will briefly conclude on the third, in which I will expose some of my ideas still in the elaboration phase.

1. Mathematics is universal

Mathematics expresses itself like few other disciplines in a universal language.

Mathematics shares this characteristic with music and also with other sciences and other arts.

The places where high mathematics is done are typically frequented by women and men of all nationalities and creeds, united together by the pleasure of a common enterprise; and there is nothing that can promote peace like recognizing in other human beings our same emotions and interests, thus truly feeling all brothers.

1. Mathematics is universal

But even in the most everyday places, close to our experience, such as schools of all levels, mathematics is a discipline that knows no differences in languages, skin color or traditions, and for this reason allows us to bring together girls and kids like few other topics. I am thinking in particular of the role that mathematics can play in the multi-ethnic classes that we encounter more and more frequently in our schools.

1. Mathematics is universal

With this first point we can therefore see how doing mathematics at all levels is a peaceful and pacifying activity like few others.

Moving on to the following two points, we want to state more:

not only is the internal work of mathematics an example of peaceful activity,

but mathematical thought can also contribute to peace in the external world, that is, outside mathematical or technical activity.

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2. **Mathematics is search for precision**
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longer about the second, which recalls the title of my speech,
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2. Mathematics is search for precision

That precision is essential in mathematics more than in other disciplines is certainly a shared thing.

But I want to emphasize the aspect of **search, the search for precision.**

2. Mathematics is search for precision

Because if mathematics at school is presented, as it is usually done in textbooks, as a disciplinary body already fully packaged, two serious risks are immediately run.

1. That many students become disaffected, and in too many cases completely lost, because they feel math incomprehensible or at least too difficult, in any case foreign to them.
2. On the contrary, that those students who feel at ease, perceive math as a world that is distant and separate from the real world, marked by a profound difference, namely that math is a domain of precision whereas uncertainty prevails in life.

2. Mathematics is search for precision

I suggest that all this doesn't happen if precision is achieved gradually, as a tendency to approach slowly, in a process that works very well in mathematics, but which can also be advantageously implemented in other situations, as we will see later.

2. Mathematics is search for precision

For example, let's think about the particular role of definitions. Why we define the notions we are dealing with in mathematics, without margins of ambiguity? Because, as is well known, in this way we ensure that rigorous reasoning can be applied to them and thus arrive at formulating statements of maximum reliability, as are the theorems, the most elementary ones as well as the most hidden ones.

2. Mathematics is search for precision

We know all this very well.

But perhaps we are not always able to derive from this all the educational advantages that can be gained.

It is constantly said, in common sense as well as in official documents, that mathematics teaches thinking, and indeed it is intended that this is the main function of its teaching. But in order for this not to be a fruitless slogan, we need to rethink in depth how we teach mathematics.

2. Mathematics is search for precision

Let me explain.

In the first place, the definitions should not be given a priori, to then see all the consequences that arise, but they should be the culmination of a path in which more advanced levels of rigor and safety are achieved little by little. In other words, mathematics should as much as possible be presented as something to be built and conquered rather than as a perfect building already beautiful and done.

2. Mathematics is search for precision

Along this path a special role is played by errors (according to etymology: a deviation from the principal direction of the path).

We refer mainly to Borasi for her view on the importance of errors in Math Education.

Errors allow to understand students' views and their reasoning. Errors open new opportunities for learning.

Among the errors, I want to speak here of the importance of ambiguity, which I intend as a special kind of linguistic error or difficulty.

2. Mathematics is search for precision

The search for precision is a process that can and must begin very early, right from the earliest years of school. After all, children are often the first to require great clarity as a condition for understanding.

As a first example of this, I present an activity that was carried out repeatedly in primary school (second and third grade), and which is part of a trend in the use of storytelling to convey mathematical content. Here, the story is that of "Goldilocks and the Three Bears" (by the writer Robert Southey, 1837)

Goldilocks and the Three Bears

Robert Southey
1837

or:

Proportionality with children, between storytelling and constructions

I hope many of you already know the story. Unfortunately I don't have enough time to read it.
However that’s the story, in a nutshell.

There are three bears, Papa Bear, Mother Bear and their son, Little Bear. Mother Bear is half the size of Papa Bear and the son is half the size of Mother Bear.
Each of them has an appropriately sized bowl, chair and bed.
Goldilocks is a little girl who enters their house in their absence, observes, touches, finally runs away.

The complete story (adapted for educational purpose)

"Once upon a time there were three bears who lived in a small house in the forest. There was a big daddy bear, there was a mother bear half the size, and finally there was a small bear half the size of his mother. One morning the three bears were having breakfast and Papa Bear said, «This soup is too hot. Let's wait for it to cool down by taking a walk in the woods».
So, the three bears came out of their house and went into the forest. While they were out, a beautiful blonde girl named Goldilocks came over there and, seeing the little house in the woods and wondering who lived there, knocked on the door. Nobody opened, but the little girl, who was very curious, went in anyway, seeing a beautiful table set for three.

There was a large bowl, a bowl half the size and a third bowl half the size of the latter. Goldilocks tasted the soup in the large bowl: «Oh! It's too hot, and the bowl is too big and heavy for my little hands», she exclaimed; then he tasted the soup that was in the bowl half the size : "Oh! It's too cold! ' There was only the small bowl left. Goldilocks sipped the soup from the latter too: «Oh! This is fine!» and satisfied, she emptied the small bowl completely.
Then he went into another room where he saw three chairs. There was a big chair, a chair half the size, and a third chair half the size of the latter. Goldilocks sat down on the big chair: «Oh! It's too hard!» she said. Then he sat down in the second chair: «Oh! It's too uncomfortable!» she exclaimed, then sat down in the little one: «Oh! This is fine!» But the chair broke and Goldilocks hurt his foot when he fell and he felt the need to lie down for a while.

She then entered another room. There she saw three beds: there was a large bed, one half the size, and one half the size of the latter. Goldilocks lay down on the big bed: «Oh! This is too hard!» she said. Then she tried the bed that was half the size: «Oh! This is too soft!» she exclaimed, then he lay down on the little bear's bed and said, «Oh! This is really good!» and comfortably lying on the bed, the baby fell asleep.
Meanwhile the three bears returned home; they looked at the table and big Papa Bear said with his strong voice: «Someone has tasted my soup!» Mother bear, half the size, said: «Someone has tasted my soup!» Finally, the little bear exclaimed: «Someone tasted my soup, and drank it all!» The three bears entered the other room. Big Papa Bear said: «Someone sat in my chair!» Mother bear, half the size of him, said: «Someone has sat in my chair!» The little bear exclaimed: «Someone sat on my chair and broke it!»

Here the three bears entered the bedroom. Big daddy bear said, «Someone has lain down on my bed!» Mother bear, half the size of it, said: «Someone has lain down on my bed!» Finally the little bear shouted: «Someone has lain down on my bed, and here she is!»
That shrill scream woke Goldilocks with a start. When the girl saw the three bears in front of her, more frightened than ever, she jumped off the cot, ran out of the room, jumped out of the low window and ran away into the forest, as fast as her legs had never made her run. She ran to her mother and once she got home she told her everything that had happened but her mother didn't believe her.

Children, help little Goldilocks to clarify everything that has happened to her. Try to represent the three bears, the three bowls, the three chairs and the three beds so that Goldilocks' mother can believe her !!

Aspects of the story

1. The text has been rewritten, with respect to the original, to be adapted for didactic needs.
2. The problematic situation is posed by Goldilocks herself to the pupils.
3. The story does not contain an explicit mathematical question

Half/Double

Proportionality

Bigger /smaller

Measure units

The three dimensions

The project

Phase 1
Presentation of the story

Phase 2
Guided discussion

Phase 3
Graphic representations

Phase 4
Measures

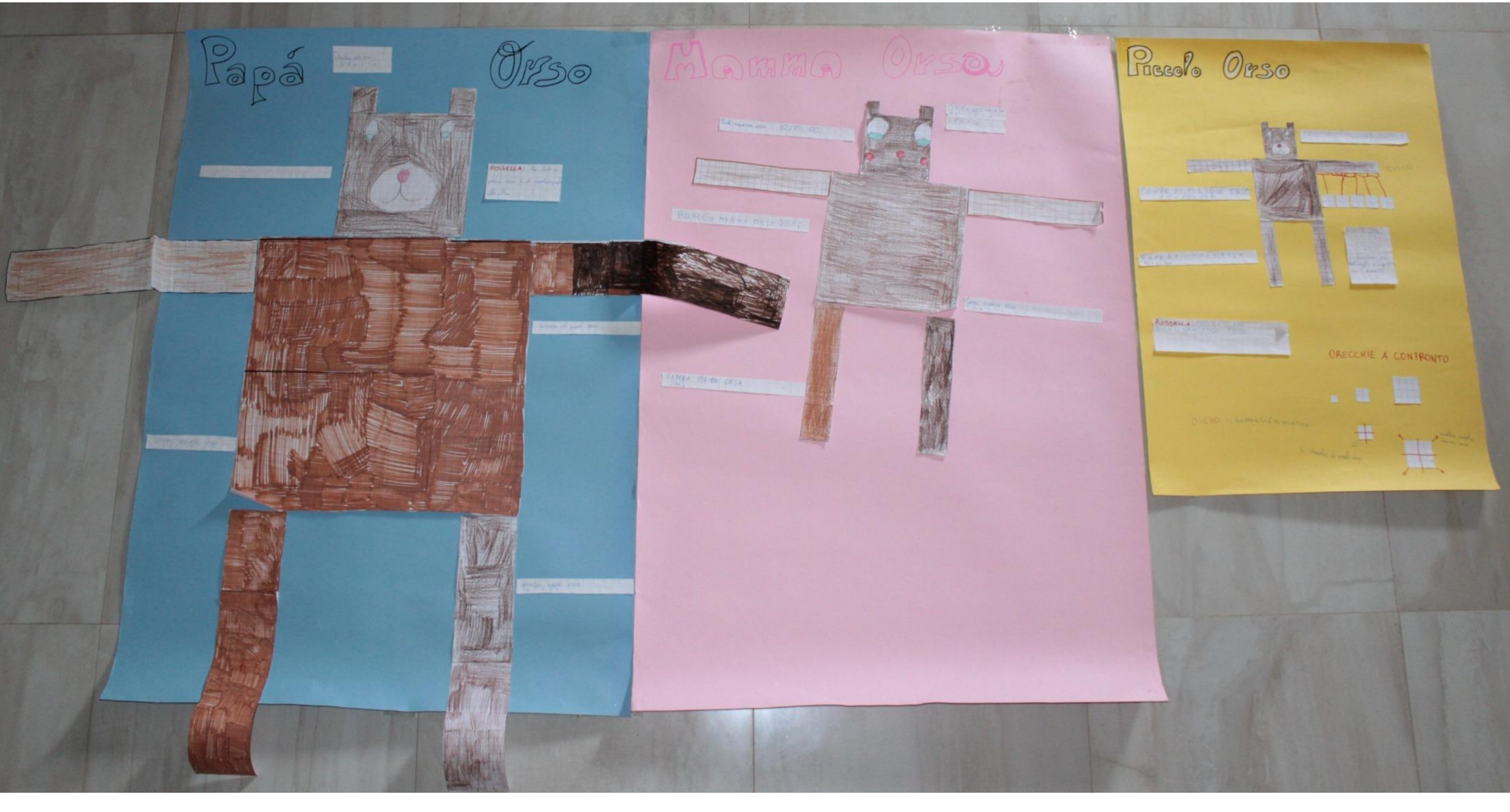
Phase 5
Discovery of the third dimension

Phase 6
Verification

The children and us

The three bears

2 DIMENSIONS: Height and width



Check

Let’s make shirts for the three bears

«Life is missing!»

Mattia's idea

3 DIMENSIONS:
Height, width and depth

What is depth?

"It's when you take a dip and see how deep the sea is.»

"It's when you enter a tunnel, the dark one. And you are afraid it will never end! "

"It is when you dig and make a ditch in the sand and enter it. That's deep! "

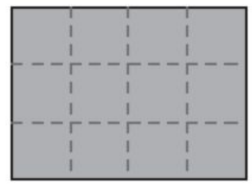
Large and small cubes

2. Mathematics is search for precision

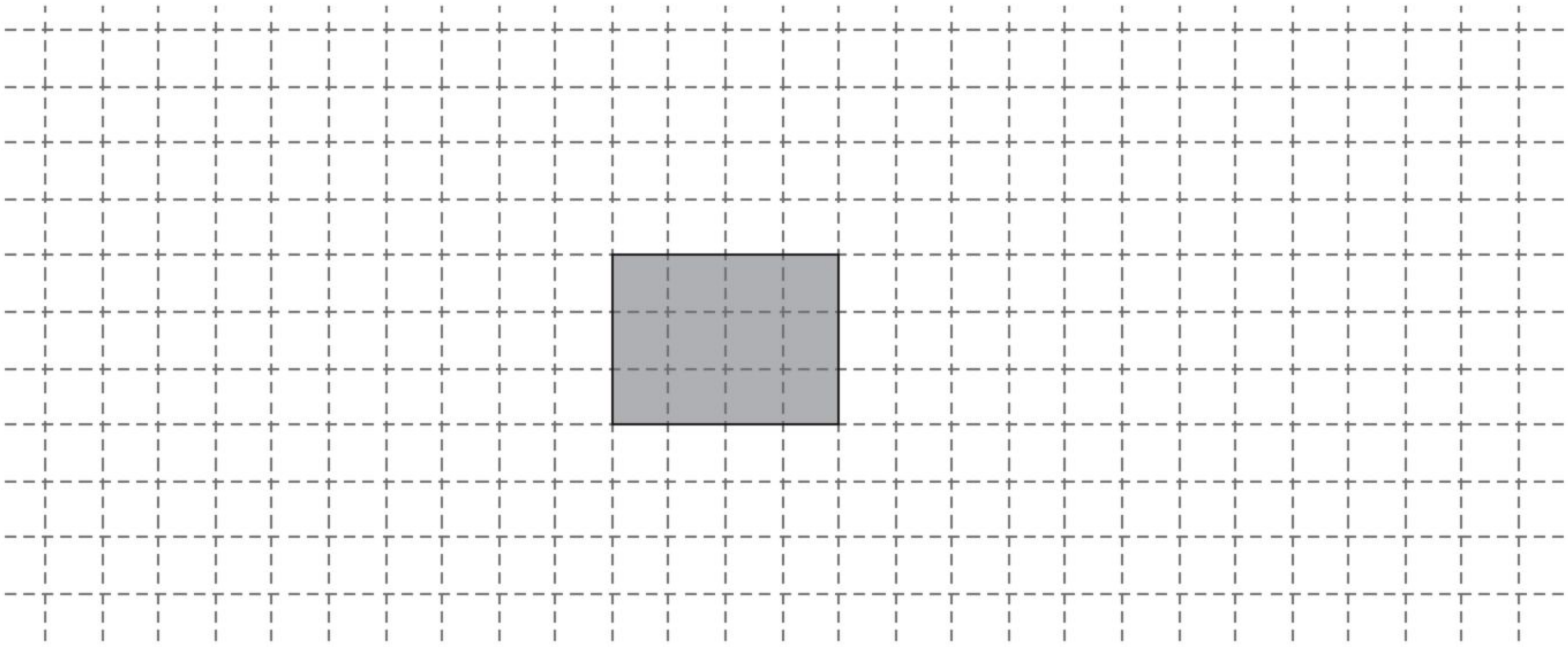
In the following slides I show some questions that were submitted to students of levels from fifth to eighth in the context of an analysis on mathematical students' performances in Italy (INVALSI tests).
We see in them an improper use of fractions, just as in the case of the three bears

Fifth grade - 2012

D11. **Il rettangolo che vedi di seguito corrisponde a $\frac{1}{4}$ di una figura.**

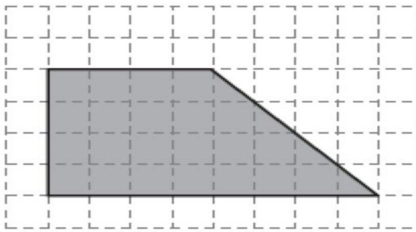


Disegna nello spazio qui sotto una delle possibili figure da cui il rettangolo è stato ritagliato.

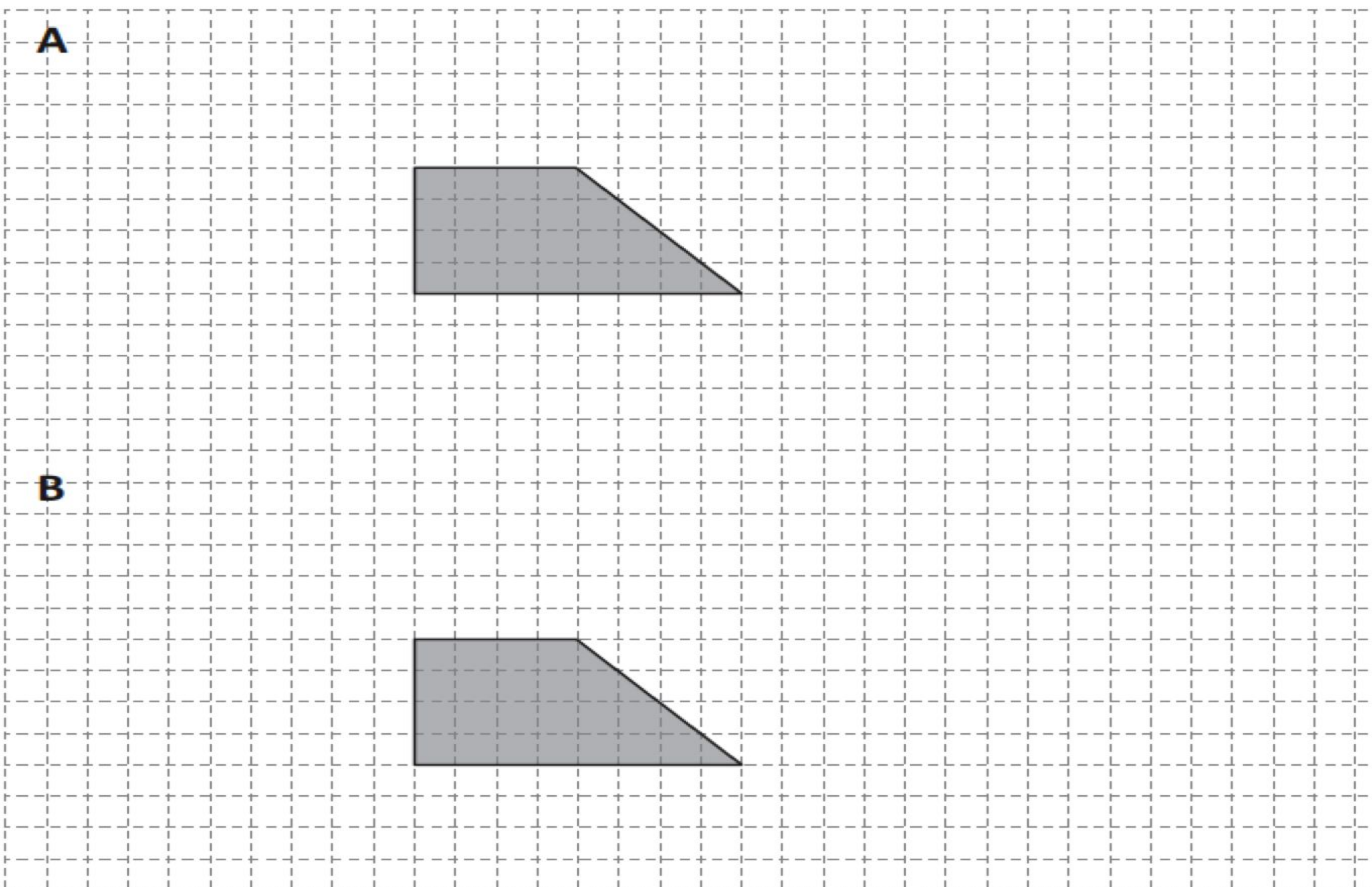


Sixth grade - 2012

D20. **La figura che vedi di seguito corrisponde ai $\frac{3}{4}$ di una figura più grande.**



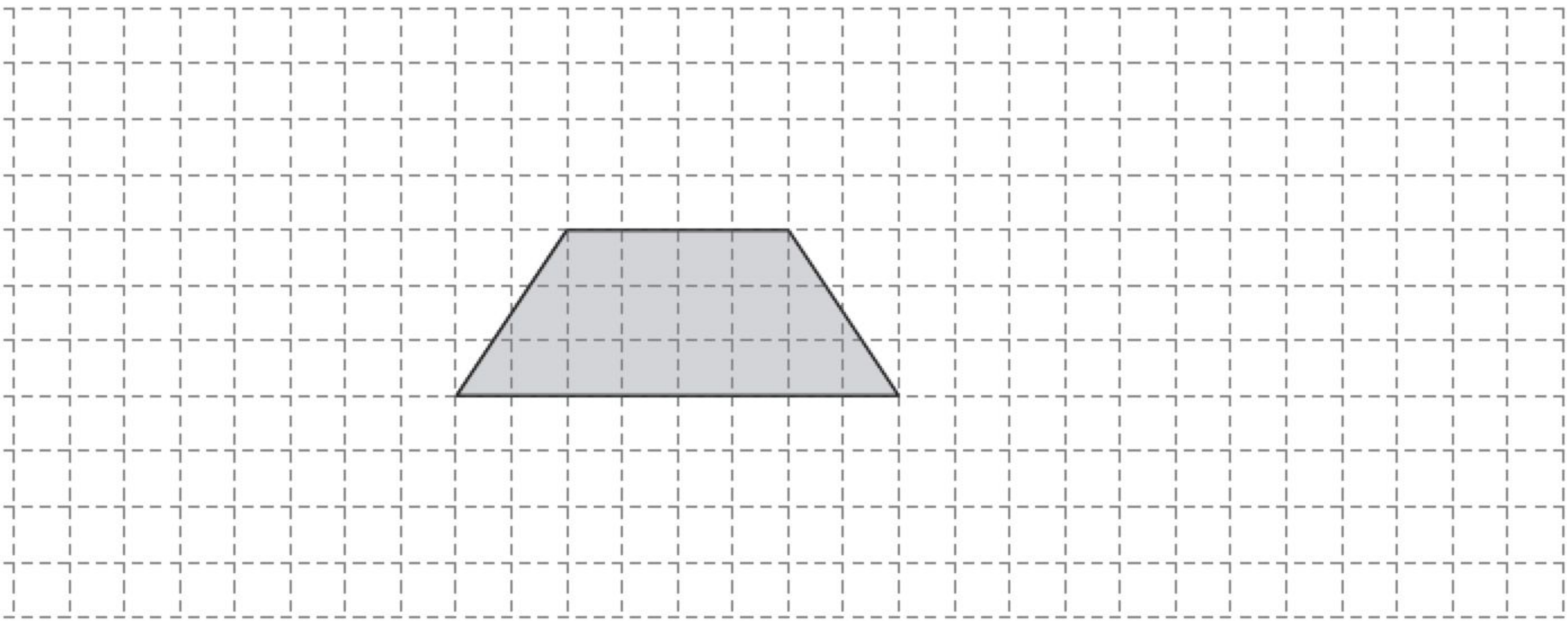
Disegna due delle figure, una nello spazio A e una nello spazio B, da cui la figura che vedi sopra può essere stata ritagliata.



E18. **Il trapezio che vedi sotto è stato ritagliato da una figura *F* più grande. Il trapezio è i $\frac{3}{4}$ della figura *F*.**

Eighth grade - 2012

Disegna una delle possibili figure *F* da cui il trapezio è stato ritagliato.



2. Mathematics is search for precision

Another typical case of misuse of words in geometry, is when some names depend on orientation: base and height of a rectangle, oblique side of a trapezoid, and so on, which generate insidious and resistant misconceptions reinforced by stereotyped drawings.

See also the confusion between vertical lines and perpendicular lines, both linguistic and conceptual.

Let's examine the special case of the square/rhombus. While we know that every square is a rhombus, a widespread misconception argues that this is not the case, but not so much because the rhombus does not have equal sides, rather because the rhombus "rests on the tip".

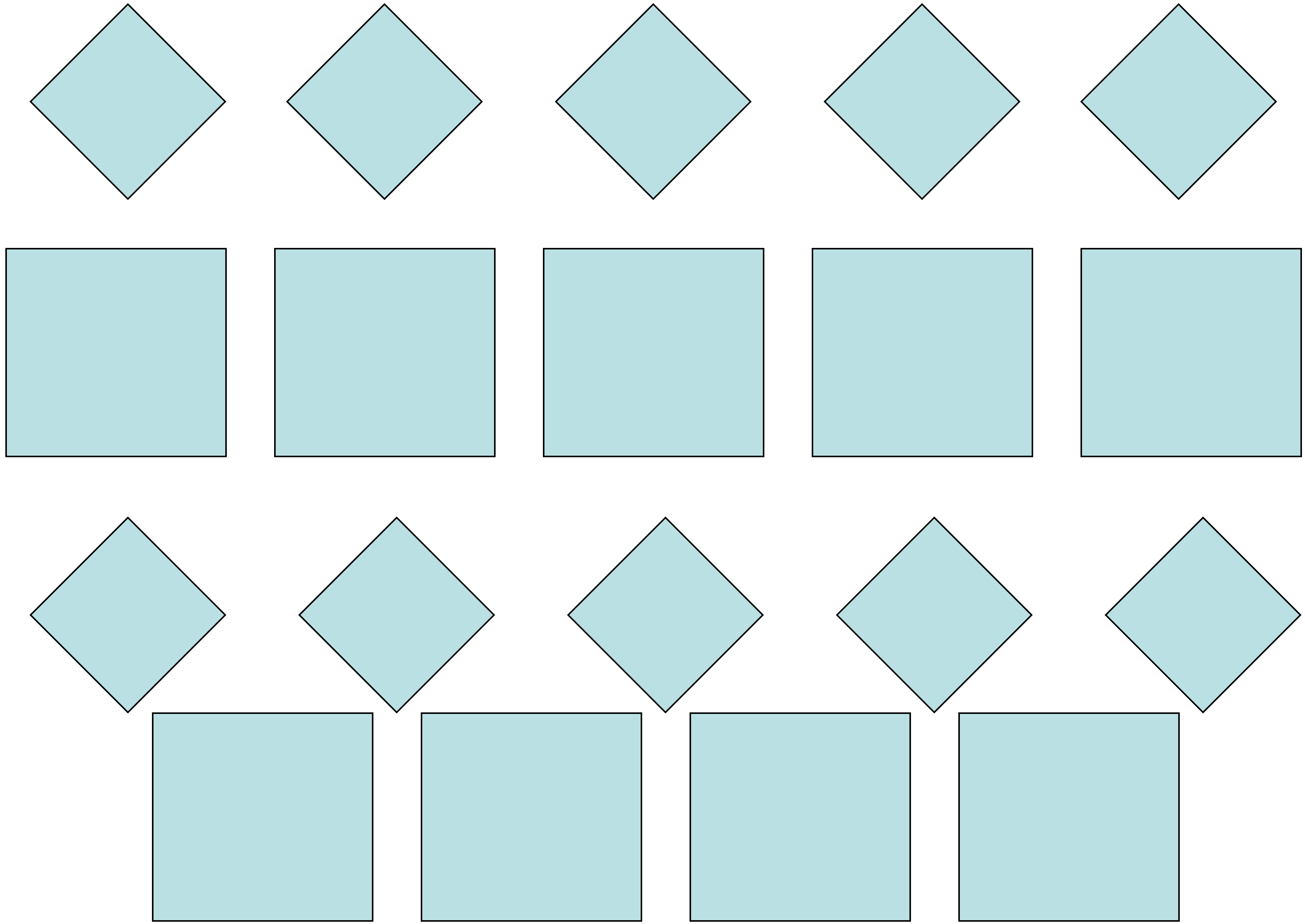
An episode.

In a recent exhibition dedicated to Escher’s paintings, many explanatory panels had been prepared, especially aimed at school groups.

In some paintings there were alternations of squares with sides parallel to the edges of the frame and squares with the sides oriented at 45°with respect to them.

Well, the panels systematically described them as squares and rhombuses.

An intolerable mistake for a mathematician.



Not that rhombuses are not squares, but qualifying a quadrilateral as a square or rhombus in dependence of its spatial orientation, and not of their geometric properties (equality of sides and/or angles), is a serious misconception! Yet it is very common use.

In my opinion, this is a typical case in which the error is certainly serious, but it is an error of inappropriateness, and consists in the use of too colloquial language unsuitable for a public context and having clear educational purposes.

I argue that very often the errors of students are of this nature, and should be classified as such.

This is to remember that in mathematics rigor is not something acquired once and for all but rather it is a never-ending journey

2. Mathematics is search for precision

The following example concerns a word problem, say, for secondary school, sixth to eighth grade.
Let's give a rapid look.

WHICH MEAN?

During the first half of a car trip, you proceed at a constant speed of 60 km/h.

For the other half, at a constant speed of 100 km/h.

What is the average speed?

It is very likely that the first answer is the following:

$$\frac{(60+100)km/h}{2} = 80km/h$$

that is the arithmetic average between 60 km/h and 100 km/h, considering that the arithmetic average is the best known and most used average.

With a careful reading of the text, however, anyone can observe that the expression "half of a journey" lends itself to at least two interpretations: half way or half travel time.

Open problem situation in which not all the data of the problem are provided, but space is left for the interpretations and consequent decisions of those who have to solve the problem.
One might think that the problem is wrong or that the trace is wrong; on the other hand, whenever the text with which a problematic situation is described is not well delineated, a space opens up for the student where he is called to analyze the situation from several points of view, make hypotheses and study its compatibility with the situation date.
The problem is an example of a good problem, in the sense of a problem in which the text does not immediately lead to the application of a procedure, but requires mediation which in turn requires reasoning, an evaluation. On the contrary, *bad problems* are those that can be solved with procedures that require only the application of known rules and the carrying out of calculations; it would be preferable to talk about *application exercises*.
Moreover, ours is a plausible problem, considering that this type of question is frequently asked in everyday life.

***First interpretation*:**

Half the journey = Half the journey time

This is equivalent to saying that

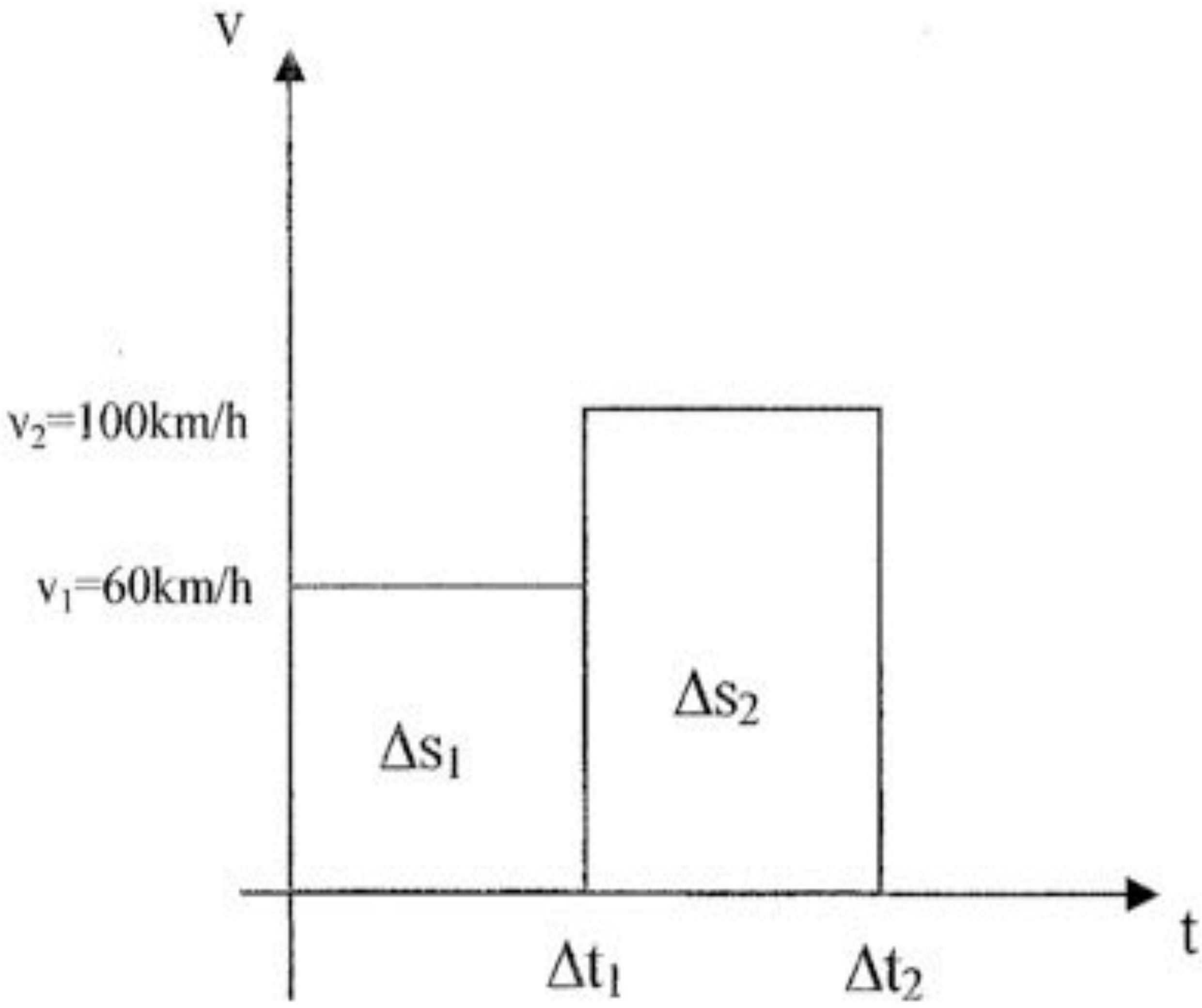
$$\Delta t_1 = \Delta t_2$$

Since the average speed is given by the total distance traveled divided by the total duration of the journey, we have:

$$v_m = \frac{\Delta s}{\Delta t} = \frac{\Delta s_1 + \Delta s_2}{\Delta t_1 + \Delta t_2} = \frac{v_1 \cdot \Delta t_1 + v_2 \cdot \Delta t_2}{\Delta t_1 + \Delta t_2} = \frac{\Delta t_1 (v_1 + v_2)}{2\Delta t_1} = \frac{v_1 + v_2}{2}$$

which is the arithmetic mean of the two speeds; we find the result proposed initially.

In this case v_m *is the (constant) speed that should be kept for the overall duration of the journey to cover the same space at the same time.*



v
v_2=100km/h
v_1=60km/h
Δs_1
Δs_2
t
Δt_1
Δt_2

***Second interpretation*:**

Half the journey = Half the distance traveled

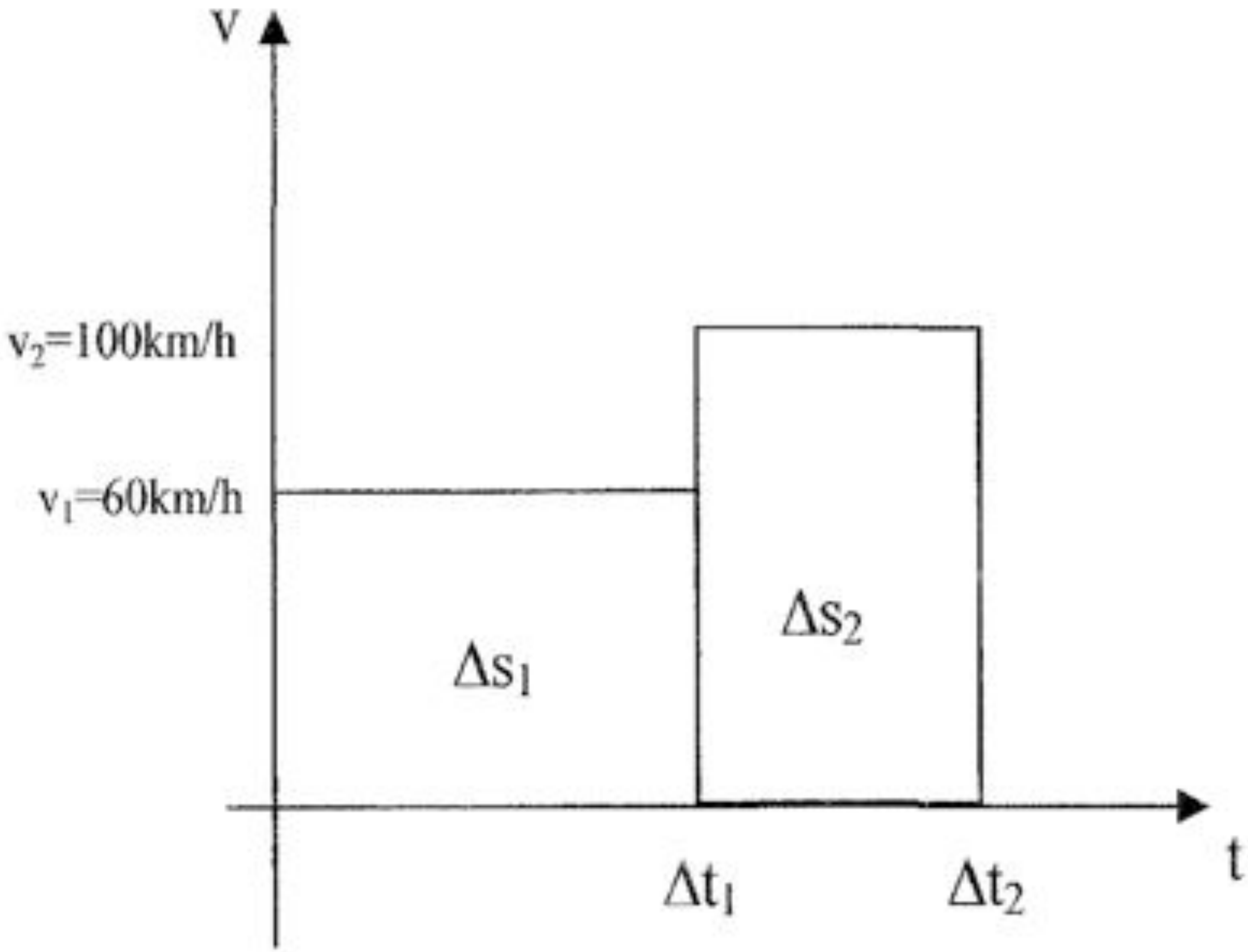
This is equivalent to saying that $\boldsymbol{\Delta s_1 = \Delta s_2}$.
In this case the average speed will be given by:

$$v_m = \frac{\Delta s_1 + \Delta s_2}{\Delta t_1 + \Delta t_2} = \frac{2\Delta s_1}{\dfrac{\Delta s_1}{v_1} + \dfrac{\Delta s_2}{v_2}} = \frac{2\Delta s_1}{\Delta s_1(\dfrac{1}{v_1} + \dfrac{1}{v_2})} = \frac{2}{\dfrac{1}{v_1} + \dfrac{1}{v_2}}$$

and the harmonic mean of the two speeds is obtained.
The numerical solution of the problem, in the second interpretation, is:

$$v_m = \frac{2v_1v_2}{v_1 + v_2} = \frac{2(60km/h)(100km/h)}{160km/h} = 75km/h.$$

This time v_m *is the (constant) speed that should be kept in traveling the total space so that the trip has the same overall duration.*


v
v_2=100km/h
v_1=60km/h
Δs_1
Δs_2
Δt_1
Δt_2
t

2. Mathematics is search for precision

I'll present you another example as the report of the conclusion of an activity conducted in a primary school (2nd grade), of exploration of arithmetic regularities (with an early start to algebra and argumentation).
All the activity was conducted in a cooperative form, and concerns what is obtained by adding three consecutive natural numbers.

Let's play with numbers

Play with numbers in consecutive sequences to discover regularities

Teacher
Carmela Pagnozzi

Mentor
Prof. Roberto Tortora

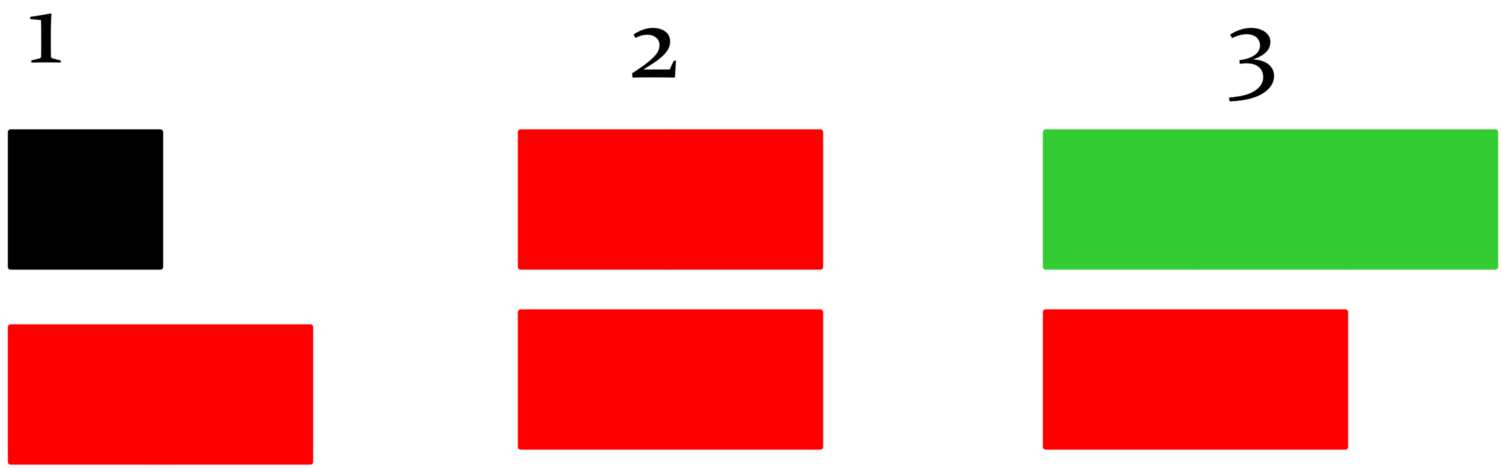
Parents, now it's your turn!

Give us three consecutive numbers!

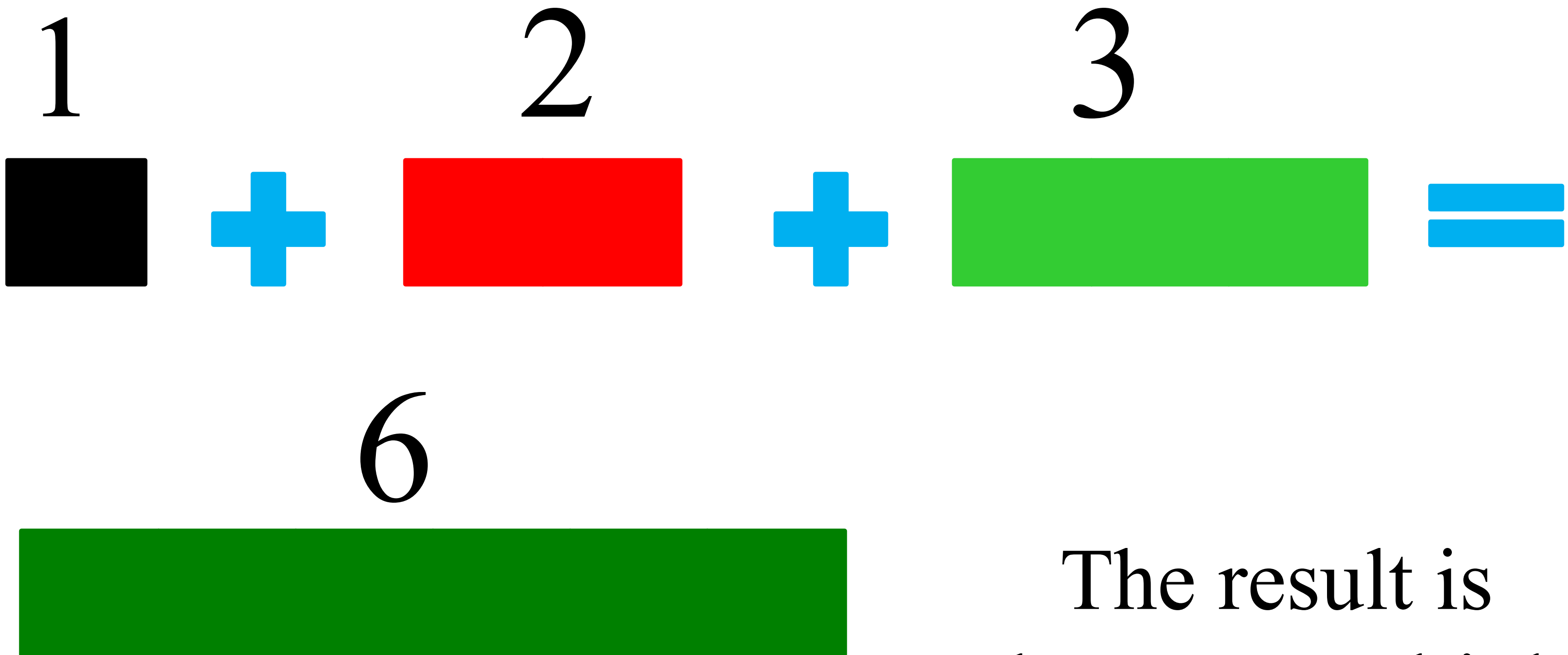
Let’s add them up together

What is the relationship between the results obtained and the three starting numbers?

Regularities



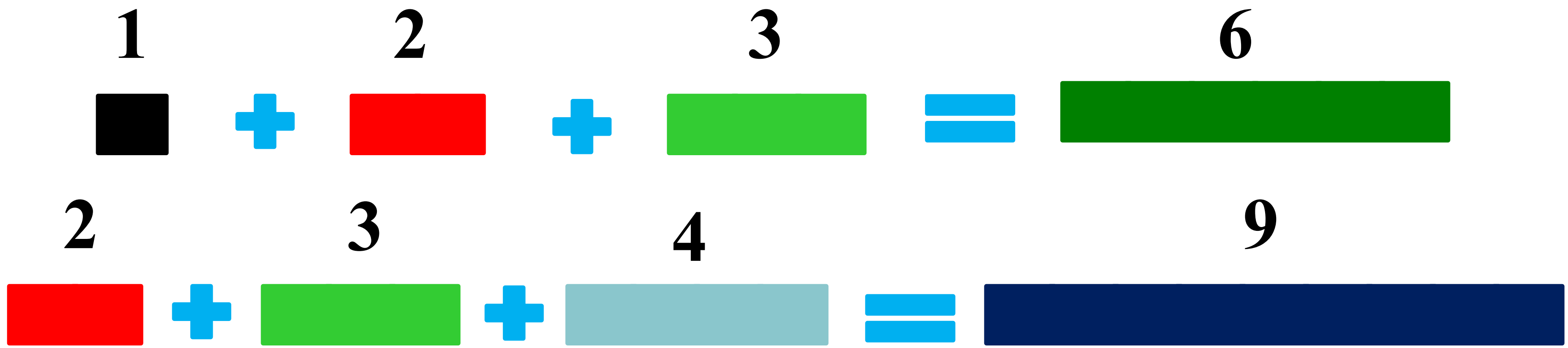
If we remove a unit from the last number and add it to the first, we get 3 equal numbers



$$6:3 = 2$$

$$6:2 = 3$$

The result is always a multiple of 3 and of the central number



If the middle number is even
the result is also even
If the middle number is odd
the result is also odd

Other regularities

1-2-3	3-4-5	5-6-7	7-8-9	9-10-11
6	12	18	24	30

The difference between each sum and the following is always constant, in fact, it is +6, which is a multiple of 3

2. Mathematics is search for precision

There are several types of ambiguity. A typical case is when the same word is used with different meanings within a colloquial or a literate register (example: angle). Many researches focus on this point, mainly based on functional linguistics.
Here we deal with a different case, that is the variation of the meaning of a word, corresponding to the enlargement of a mathematical concept.
The word we examine is the adjective "consecutive" in an arithmetic/algebraic context. (Of course there are other meanings in other contexts, see e.g. in geometry the notion of consecutive segments).

2. Mathematics is search for precision

I have presented before a simple activity with consecutive numbers, for primary school.

There, of course, no problem appears concerning the meaning of consecutive. However, from their drawings, we see that pupils «know» that two numbers are consecutive if the latter is equal to the «former plus one». If we want to procede in the direction of problematize the notion, it is possible to make a new step, asking pupils to add up three consecutive even (or odd) numbers, where, of course, consecutive means «plus two».

2. Mathematics is search for precision

But rather I prefer to show you a different, and more difficult, activity with consecutive numbers, that we have experimented with secondary students as well as with future elementary teachers.

Take four consecutive numbers.

Multiply the two middle numbers by each other, then multiply the first one by the last one and calculate the difference between the two products.

Repeat the exercise several times, using different numbers.

Do you observe any regularity?

1) The first "natural" meaning of the term "consecutive" corresponds to "to be immediately subsequent to", like in "Monday and Tuesday are consecutive days". In mathematical words, an order relation is involved.

So, to say that two elements a and b are consecutive, we have to write: "$a < b$ and there is no element c such that $a < c < b$", a quite complex sentence from a logical point of view.

2) The other meaning of consecutive can be expressed as: "Two numbers are consecutive if the second one is obtained adding one unity to the first one, that is if

$b = a + 1$".

Unfortunately, the two above meanings coincide for natural numbers, but not for other kind of numbers:

- either in the sense that the first meaning disappears (for rational numbers a and b, it is meaningless to say that they are consecutive while it makes sense to say that $b = a + 1$);
- or in the sense that the second possibility vanishes (two even numbers a and b can be consecutive but it can never happen that $b = a + 1$);
- or in the sense that both conditions are acceptable but bear different meanings (e. g. in the case of rational numbers truncated to, say, the second decimal place).

In any case the two properties keep their own autonomous importance.

The solution to the problem typically develops as follows:

a) Many numerical verifications
b) The emergence of the right conjecture: “The difference is always 2”
c) Attempts to prove the conjecture.

In this phase no doubt raises about the meaning of “consecutive numbers”.

But the algebraic translation of “four consecutive numbers” can vary, from the “best one”

$$a,\ a+1,\ a+2,\ a+3$$

to other solutions, like the use of 4 consecutive letters (a, b, c, d) accompanied or not by the necessary additional conditions: $b = a+1$, $c = b+1$, $d = c+1$

Typical steps after the first version of the problem:

1. Take four consecutive even numbers; etc. (like above)

Typical steps after the first version of the problem:

1. Take four consecutive even numbers; etc. (like above)
2. Take four consecutive odd numbers; etc.

Typical steps after the first version of the problem:

1. Take four consecutive even numbers; etc. (like above)
2. Take four consecutive odd numbers; etc.
3. Take four numbers having constant distances to each other; etc.

Typical steps after the first version of the problem:

1. Take four consecutive even numbers; etc. (like above)
2. Take four consecutive odd numbers; etc.
3. Take four numbers having constant distances to each other; etc.
4. Take four consecutive non integer numbers; etc.

Four consecutive decimal numbers

It is a way of provoking the students.

Their first attempts are with truncated decimals.

…

Four consecutive decimal numbers

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The result is twofold:

- The comprehension of the density of rationals

Four consecutive decimal numbers

It is a way of provoking the students.

Their first attempts are with truncated decimals.

…

The result is twofold:

- The comprehension of the density of rationals
- A generalized formula for arithmetic progressions:

Given four consecutive terms of an arithmetic progression $(a, a+d, a+2d, a+3d)$, we get:

$$a(a+3d) - (a+d)(a+2d) = 2d^2.$$

I have collected here a few other problems containing various forms of ambiguities

IMMIGRANTS

There are 10 boys and 15 girls in a class.

It is known that 3/10 of males are non-EU and 4/15 of females are non-EU.

What is the overall ratio of immigrants to the total number of children in the class?

Bread and thought

*(*taken from **chapter 4** of the book ***The Man Who Counted***, by **Malba Tahan**, alias **Júlio César de Mello e Souza***)*

(A sort of short version, containing just the mathematical question)

Three travelers stop to rest: the first has three loaves, the second five and the third none. They put their bread together and decide to divide it equally. The third wayfarer takes eight gold coins out of his pocket, begging the other two to divide them equally, as a reward for their loaves. How should the 8 coins be divided?

Bread and thought (The story in full)

Three days later we were approaching the ruins of a small village called Sippar, when we saw, lying on the ground, a poor wanderer covered in rags who looked badly injured. He was in a pitiful condition. We set out to help him and later he told us the story of his tragedy.

His name was Salem Nasair and he was one of the richest merchants in Baghdad. A few days earlier, returning from Basra and heading for el-Hilleh, his large caravan had been attacked and robbed by a band of Persian nomads and nearly all of his comrades had been killed. He, the master, had miraculously managed to save himself by hiding in the sand among the inanimate bodies of his slaves.

When he had finished the story of his misfortunes, he asked us in a trembling voice: “Don't you by any chance have something to eat? I'm starving".

"I have three loaves - I replied".

"I have five - said the Man Who Counted".

“So - said the Sheikh I beg you to share your loaves with me. I propose a reasonable exchange. I will give you eight gold coins for bread as soon as I get to Baghdad. " And so we divided the loaves between us.

The next day, late in the afternoon, we entered the famous city of Baghdad, the Pearl of the East. Crossing a crowded and noisy square, we were blocked by the passage of a sumptuous group at the head of which rode, on an elegant chestnut, the powerful vizier Ibraim Maluf. Seeing Sheikh Salem Nasair in our company, he stopped his brilliant entourage and questioned him: “What happened to you, my friend? Why do you get here in Baghdad so battered, in the company of these two foreigners? »

The poor Sheikh told him in detail what had happened on the journey, extensively praising us.

"Reward these two foreigners immediately" ordered the Vizier. She took eight gold coins from her purse and gave them to Salem Nasair saying, "I will take you to the palace right away as the Defender of the Faithful will surely want to be informed of this new affront by the Bedouin bandits, who dare to attack our friends and plunder. a caravan on the territory of the Caliph ".

At this point Salem Nasair told us, “I take leave of you, my friends. However, I would like to thank you once again for your help and, as I promised, to compensate you for your generosity. And, turning to the Man Who Could: "Here are five gold coins for your five loaves". Then to me: "And three to you, my friend from Baghdad, for your three loaves".

To my surprise, the Man Who Counted respectfully raised an objection. “Forgive me, Sheikh! But this subdivision, which seems simple, is not mathematically correct. Since I have given five loaves, I must receive seven coins. My friend who has given away three loaves, must only receive one ”.

“For Muhammad! exclaimed the vizier, keenly interested. - How can this foreigner justify such an absurd claim? "

The Man Who Counted approached the minister and said: "Allow me to show, O Vizier, that my proposal is mathematically correct. On the way, when we were hungry, I took a loaf of bread and divided it into three parts.

We each ate one. So my five loaves got us fifteen pieces, didn't they? My friend's three loaves added nine pieces, for a total of twenty-four pieces. Of my fifteen, I consumed eight, so I actually gave up seven. Of his nine pieces my friend also ate eight and so his contribution was only one. My seven pieces and my friend's one make up the eight that went to Sheikh Salem Nasair. Therefore it is right that I receive seven coins and my friend only one»

The Grand Vizier, after highly praising the Man Who Counted, ordered that he be given seven gold coins and one to me. The mathematical proof was logical, perfect, irrefutable. But, however correct, the subdivision did not please Beremiz who, turning to the surprised minister, continued: “This division, seven for me and one for my friend, is, as I have proved, mathematically perfect; but it is not perfect in the eyes of the Almighty ". And, collecting the coins again, he divided them into two equal parts, four to me and four to himself.

“A truly extraordinary man! - exclaimed the Vizier - He did not accept the division of the eight coins into five and three. He has shown that he is entitled to seven and his partner only one. But then he divides the coins into two equal parts and gives one to his friend. And he added with enthusiasm: "By the Almighty! This young man, in addition to being good and fast at math, is a good and generous friend. I want you to become my secretary today ”. "Grand Vizier - said the Man Who Counted - I realize that you have expressed, in thirty words and 125 letters, the highest praise I have ever heard. May Allah bless you and protect you for all eternity!

"The ability of my friend Beremiz allowed him to keep track of the words and letters he spoke ... We all marveled at this display of genius.

2. Mathematics is search for precision

We have seen, with many example, how precision can be pursued within mathematics.

But I want to emphasize now that this attitude to seeking ever more precision should be transferred to all fields of knowledge.

And if it certainly true that mathematics is the topic, of all, where you can go further in the search for rigor, it is however crucial to make students understand how important it is in every field to try to clarify notions, concepts, words, before making statements on them.

2. Mathematics is search for precision

Students of all ages should be encouraged to get out of vagueness whenever possible. This does not imply to neglect the needs of other, complementary, approaches to knowledge, for example literature or poetry. What is important is to keep distinct the different contexts.

In this search for precision in everyday life, logic and in particular the notion of truth have a special role.

Aristotle in *Metaphysics* states:

To say of what is that is not, or of what is not that is, is false,
while to say of what is that is or of what is not that is not, is true.

St. Thomas: *Adaequatio rei et intellectus.*

They might seem like things taken for granted, said in a convoluted form. It is not so.

Very precise restriction on the use of the words true / false.
Another restriction: the reference is only to statements of a particular type, those that today we would call atomic, both in affirmative and negative form.

The problem and the meaning of TRUTH

From **Aristotle** and **St. Thomas** to the

"Definition" of **Tarski**:

The proposition "Snow is white" is true if and only if snow is white.

Is the fact that snow is white safe or controversial?
What exactly is snow?
What are the colors?
How do you judge whether or not an object is of a certain color?

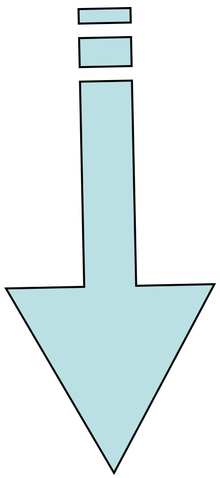
The definition of TRUTH according to Tarski

The definition of TRUTH according to Tarski

<u>The number 5</u> <u>is bigger than</u> the <u>number 3</u>

The definition of TRUTH according to Tarski

The number 5   is bigger than   the number 3



The definition of TRUTH according to Tarski

The number 5   is bigger than   the number 3

5

The definition of TRUTH according to Tarski

The number 5 is bigger than the number 3

5 3

The definition of TRUTH according to Tarski

The number 5 is bigger than the number 3

5

3

The definition of TRUTH according to Tarski

The number 5   is bigger than   the number 3

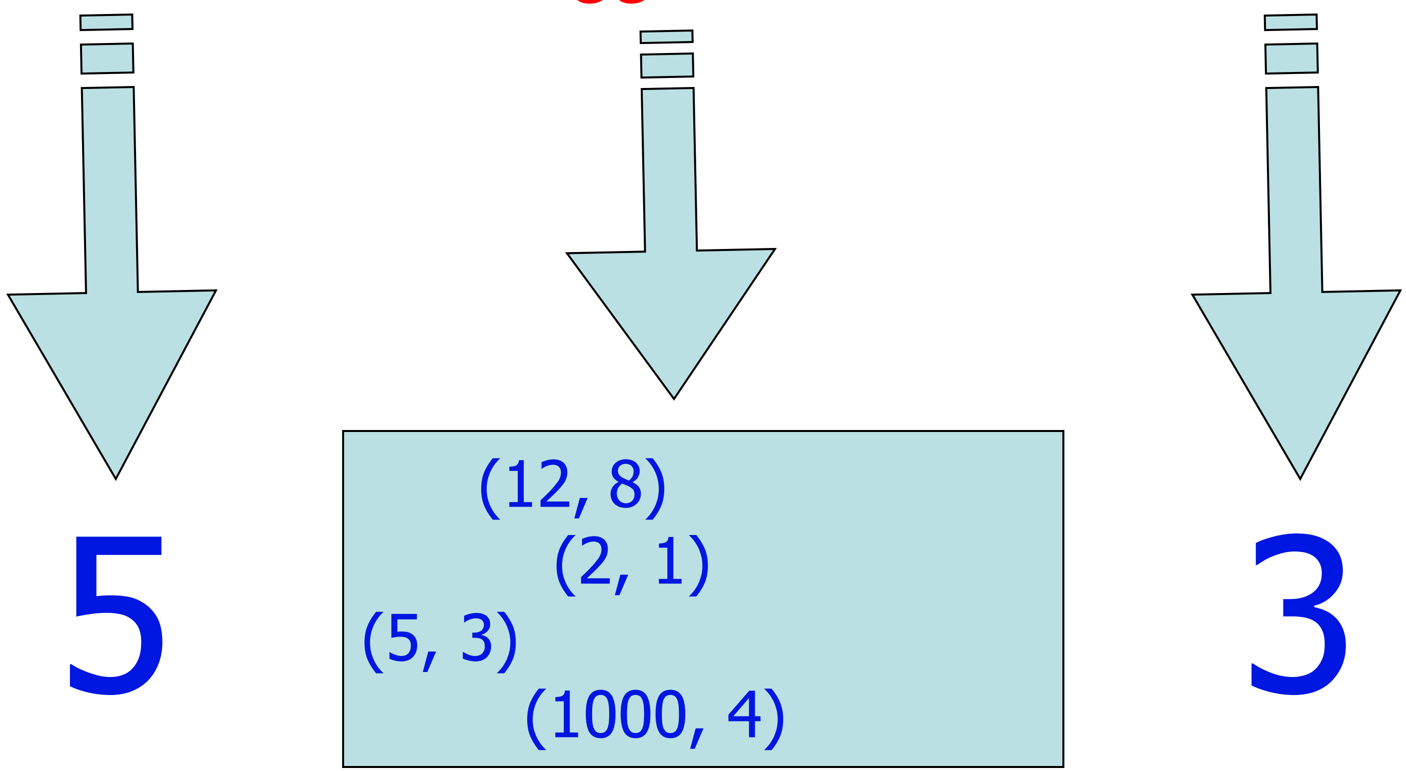
5

All pairs of numbers
where the former
is greater than the latter

3

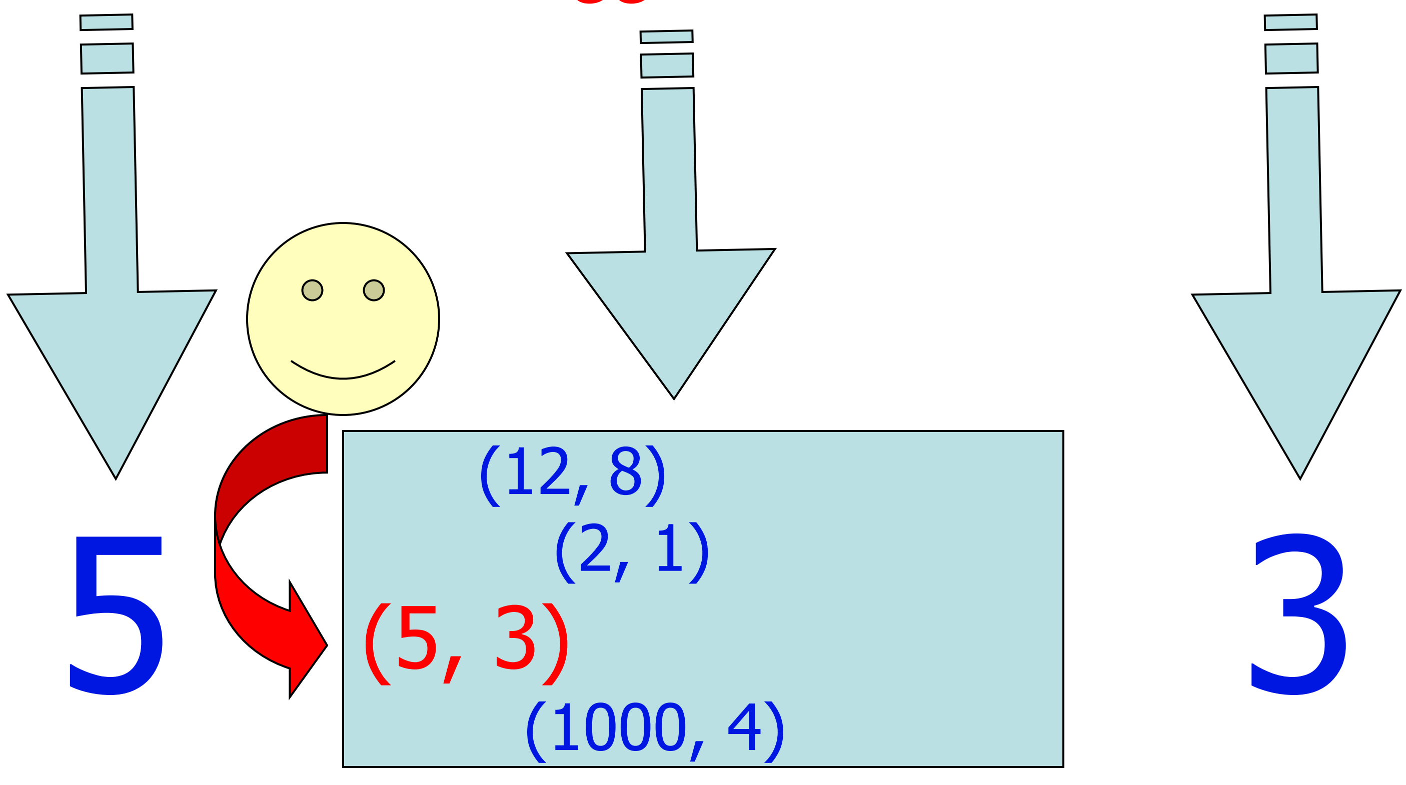
The definition of TRUTH according to Tarski

<u>The number 5</u> <u>is bigger than</u> the <u>number 3</u>



The definition of TRUTH according to Tarski

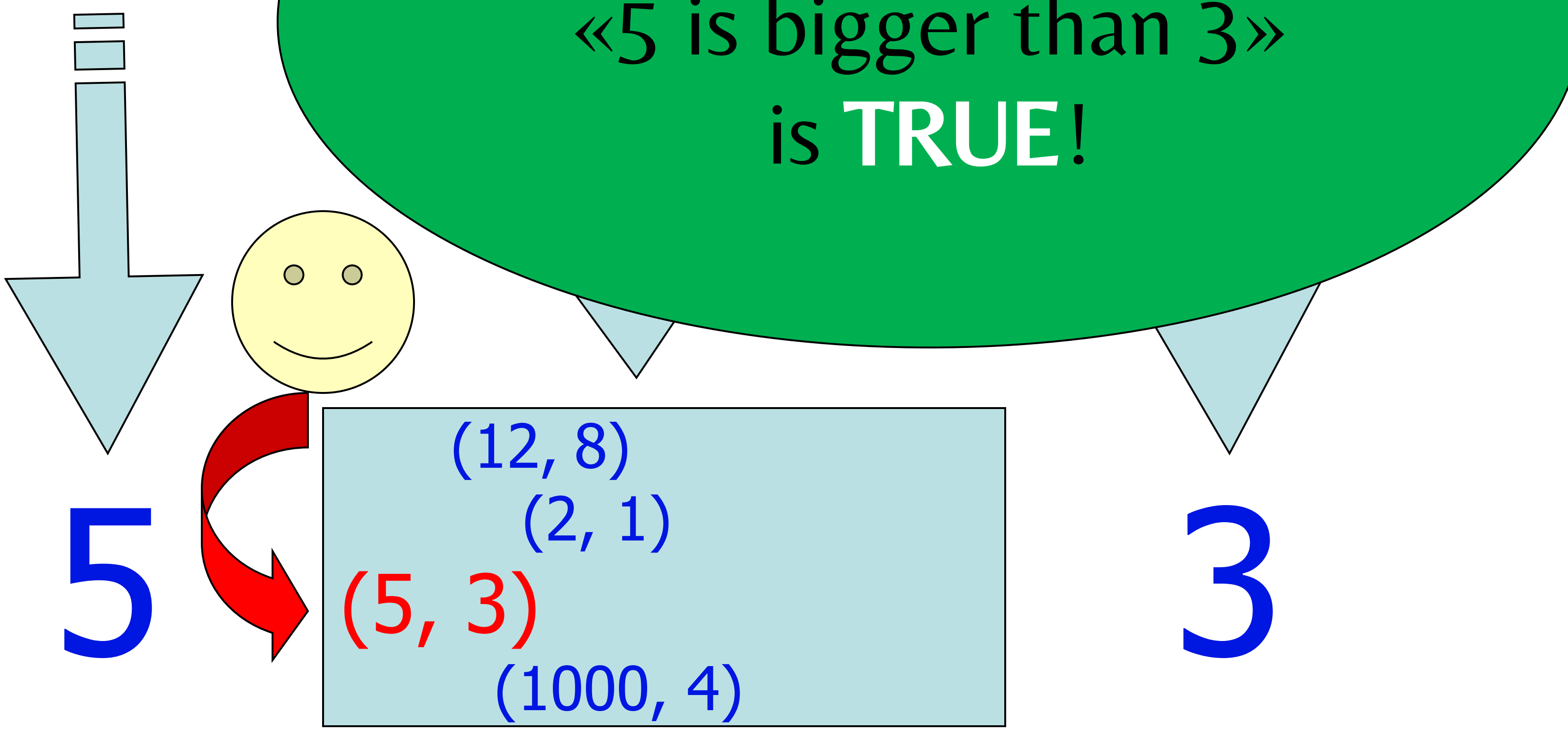
<u>The number 5</u> <u>is bigger than</u> the <u>number 3</u>



The definition of TRUTH accordi

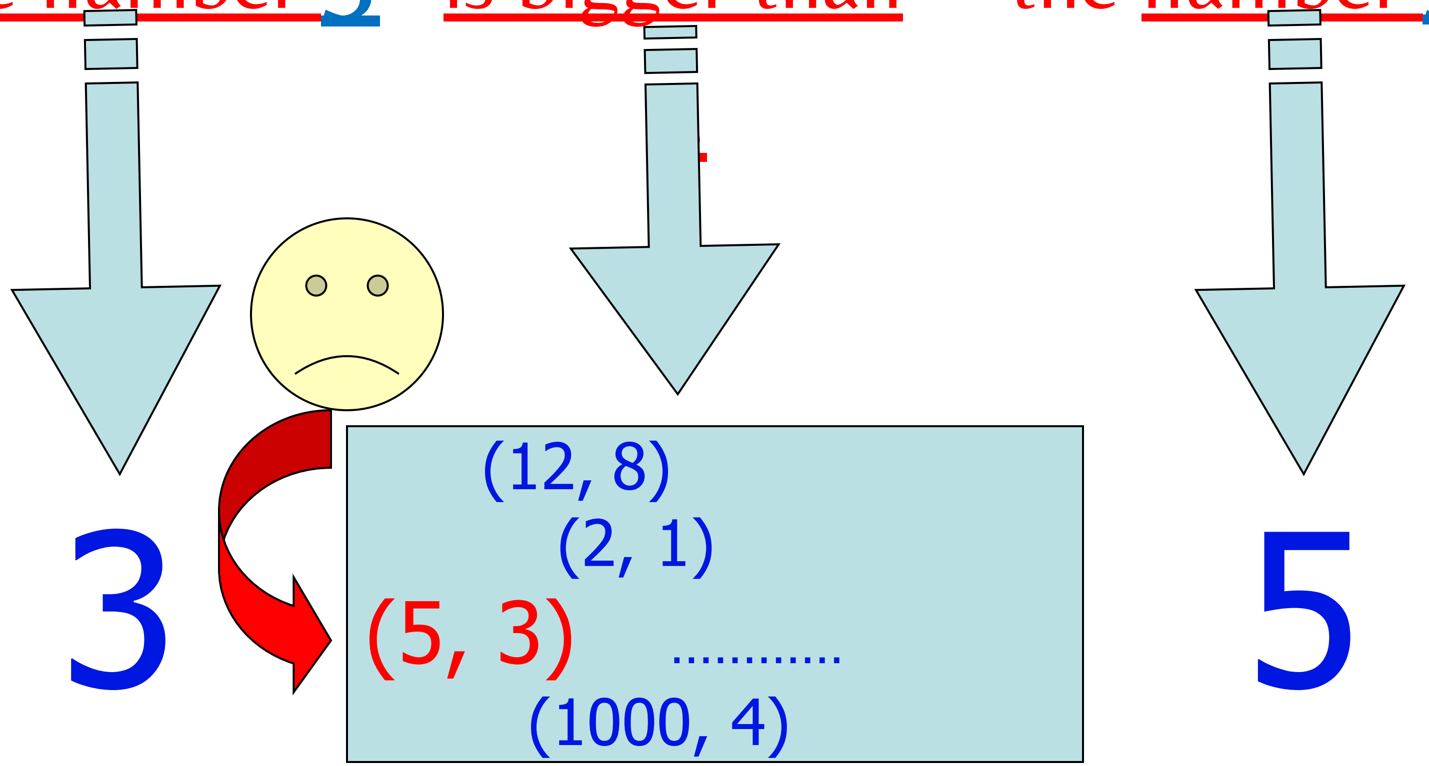
The number

The proposition
«5 is bigger than 3»
is **TRUE**!

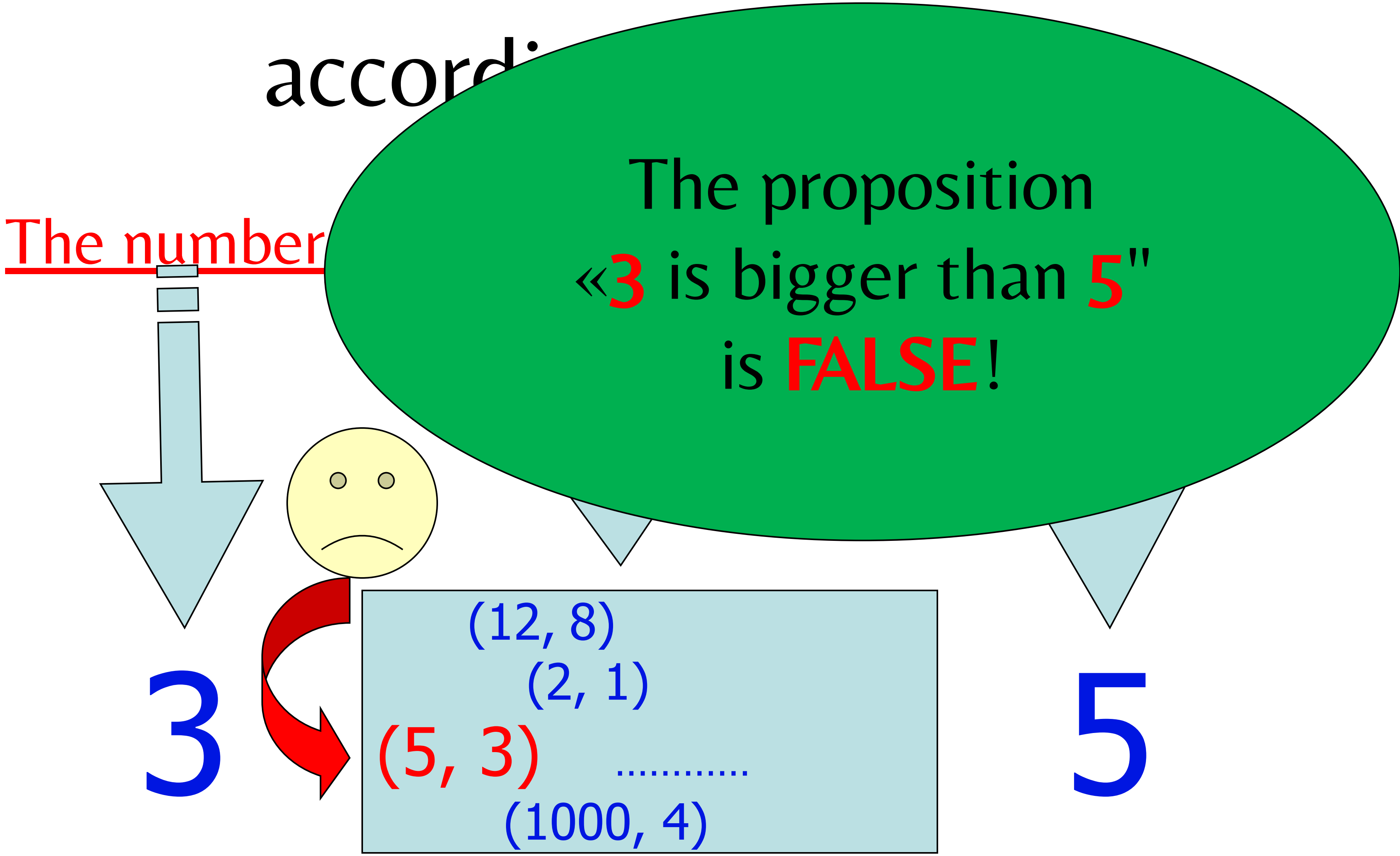


The definition of TRUTH according to Tarski

The number 3 is bigger than the number 5



The definition of TRUTH accordi



Things are simple within math. For example it is simpe to see why «5 is larger than 3» is true and «3 is larger than 5» is false.
Things are much less simple, when we get out of it.
First of all, we can observe how, behind Aristotle-Tarski's definition of truth, there are two important philosophical presuppositions.

Two philosophical presuppositions:

the first, ***ontological***, consists in the assumption that there is an objective reality;

the second, ***gnoseological***, is the assumption that this reality is knowable by us, in particular that language, as an instrument of knowledge, performs well the task of indicating and describing things of the external world

Neither is a foregone assumption; indeed, there are philosophical positions that deny one or the other of them in principle. Well, it must be said that with the supporters of these positions, dialogue is almost impossible: denying the objectivity of the external world or its knowability, means excluding at the root that we can speak of truth, at least in a sense comparable to that described above.

Often, however, denial positions are not taken as absolute, but rather are the result of unsuccessful attempts to reach some agreement on the truth of propositions of a "not simple" nature. It is from these frustrations that one often comes to argue that the truth is unattainable or even does not exist.
The latter conclusion is therefore more a declaration of surrender than a position of principle.

On this front, in my opinion, we can - we must - work in the school.

2. Mathematics is search for precision

What can be done to transfer Tarski's approach to truth into a concrete situation?

(Leibniz would have said: "Calculemus")

How can simple assertions like "John is nice" be said to be true or false?

Or, to be in line with current events, an assertion like "the Covid vaccine is effective"?

We can limit ourselves to sentences without variables, without quantifiers and without connectives.

The three following examples are atomic and singular statements:

1. The cat is in the armchair.

2. Youth unemployment in Europe has grown by 2%.

3. Jane is a good violinist.

1. The cat is in the armchair.

Easy: what is «a cat»? What «an armchair»? What does it mean «to be in»?

2. Youth unemployment in Europe has grown by 2%.

Like all sentences in which subjects and predicates are abstract notions, their interpretation offers greater problems than sentences like 1. Here we have to interpret “youth unemployment”; it is clear that behind the abstract concept there is a number, the number of unemployed young people: this is what has grown.

It remains to be understood what we mean by "young" and "unemployed". Suppose we have done it: attention, the operation can be neither simple nor safe, the notions of young and unemployed are susceptible to different meanings and it is necessary to decide which one we want to take as good or to know which the author of 2 had in mind.

But let us assume that we have done it all: mathematics after all teaches us to reason under hypotheses and thereby to separate one question from another.

Still more remains to be clarified: what time interval are we referring to? Unemployment is a number, but it depends on when it is detected. Therefore, more than a number, it is a function of time: we need two different time instants of this function and the question consists in asking whether the value of the function in the second instant is or not 2% greater than its value in the first .
Let's stop here, even if we could still discuss other things.

But coming to an ultimate conclusion is not what matters. If you don't want any room for uncertainty, then you have to work within mathematics. If, on the other hand, we do modeling, we must not be misled by the game "*Let’s find new subtle distinctions beyond those already examined*", a game that inevitably preludes to the conclusion: "*Since the sequence of necessary clarifications never ends, the entire procedure is sterile and we might as well give it up*". On the contrary, what is important is to understand that to attributing a true/false truth value to a sentence like 2, you can get as close as you can and as much as you want.

3. Jane is a good violinist.

A brief comment on sentence 3. Linguists (or some of them) call this sentence of the evaluative type: this classification would tend to remove it from the attribution of truth. But the mathematical logic approach does not require this distinction. It is evident that the judgment as to whether 3 is true or false is subjective and therefore it cannot be spoken of as objective truth.

Relative truth then? Yes, but it depends in what sense we say it.

First we interpret "Jane", but let’s assume that there are no problems with this name and that the identity of the person is known; then we interpret "violinist", and even here we assume that there are no doubts. Finally there is "good". The interpretation of good (or if you prefer, of "good violinist") is made up of a set of real individuals that we consider good (or good violinists).
Then it is done: sentence 3 is true if the individual identified by the name Jane is in this set, and it is false otherwise.
Typical situation in which the true/false evaluation of a sentence depends on the interpretation.

2. Mathematics is search for precision

What does peace have to do with this? Well, I think that a significant component of the contrasts that divide human beings at all levels lies in the mutual misunderstanding, in the fact that we give vague and different meanings to the same words. It is enough to hear one of the many talk shows on television to realize it. I therefore think that if the attitude of mathematics in search for precision were transferred at least in part to other areas of human activity, including daily ones, perhaps the reasons for misunderstanding and conflict would be attenuated at all levels.

1. **Mathematics is universal**
2. **Mathematics is search for precision**
3. **Mathematics focuses on variability and interdependence**

I will speak very briefly about the first point, easy and obvious,
longer about the second, which recalls the title of my speech,
and I will briefly conclude on the third, in which I will expose some of my ideas still in the elaboration phase.

3. Mathematics focuses on variability and interdependence

And we come to the third reason why mathematics can be useful for peace.

Here I propose a less obvious and certainly questionable reflection.

I would like to recall one of the most pervasive concepts of mathematics, that of **function**.

3. Mathematics focuses on variability and interdependence

For centuries this notion has entered mathematics and has perhaps conquered its most central position. In the field of science and technology we have long learned to think in terms of the dependence of one variable on another, and we have understood that the best way to describe the existing is precisely what is allowed by the idea of function. I am not thinking here of the function in the most abstract and modern sense of a set of pairs, but of the more basic and more familiar "real function of a real variable", especially if it is continuous or almost everywhere continuous, in short, the one that is typically visualized with graphics, that we see everywhere.

3. Mathematics focuses on variability and interdependence

Well, what is special about this notion? In my opinion, from a general cognitive point of view, thinking in terms of functions means recognizing that in the world there are no single and detached objects and qualities, but rather links between different things that vary together. In a sense, it is a question of privileging the point of view of Heraclitus (everything changes) over that of Parmenides (the fixity of being).

3. Mathematics focuses on variability and interdependence

Well, so what?

What I want to argue here is that, if it is true that in the technical field the notion of function is familiar to us, in our daily life this is not at all true.

The trouble is by no means a small one and indeed has very deep roots.

3. Mathematics focuses on variability and interdependence

We are used in our daily life to continuously express crisp judgments on single facts, other than relating variable things. For example we say that a vaccine is effective or that it isn't, that a political choice is right or wrong, and so on.

The fact is that our language was constituted in its basic structure in very distant times (more or less according to the Aristotle’s distinction between substance and accident), and therefore is typically articulated in single affirmations, in which a subject is said to possess a certain quality or perform a certain action. So, our linguistic structure is an obstacle to think differently.

3. Mathematics focuses on variability and interdependence

We find it very difficult to think that there are no "good or bad" things, "white or black", "useful or useless", etc., but rather always continuous graduations of each of these qualities. And since instead we have tools available in our language to express typically clear-cut judgments, we are forced to see and judge things so that we inevitably divide ourselves on them. We certainly know that the judgments we make are relative, but relative only means that someone thinks something is good and someone else thinks the same thing is bad.

3. Mathematics focuses on variability and interdependence

Resuming, the notion of function has indeed changed the way of thinking about the world in the scientific field, but it has not affected our way of expressing in our daily speeches and therefore of thinking, even at high levels.

Well, what can we do at school?

3. Mathematics focuses on variability and interdependence

If the idea I'm proposing is not considered completely unfounded, then one realizes that an enormous space for intervention opens in the school. In fact, one could try, within mathematics, **to acquire quickly and well the idea of the link between variables that is contained in the notion of function.**

3. Mathematics focuses on variability and interdependence

But then, immediately afterwards, tenaciously opposing the temptation to close oneself inside mathematics, we can try to transfer this dynamic vision of the world to the outside, in all fields of our experience.

3. Mathematics focuses on variability and interdependence

I think that something can be done, at all school levels.

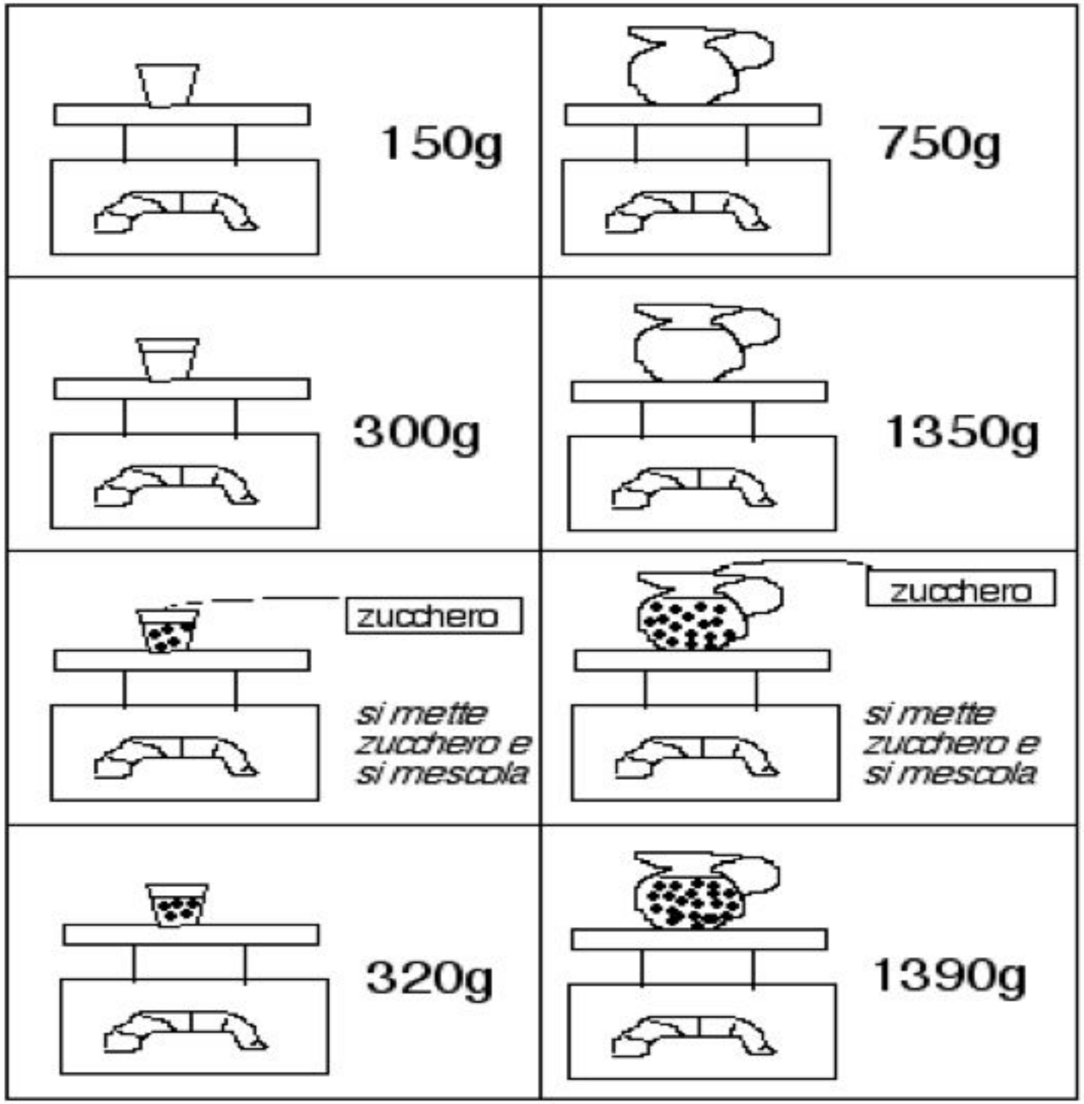
Let's see just one example of a possible activity with young children (from kindergarten to fifth grade)

Without forgetting that even the activity with the three bears is based on the comparison of variable quantities

The sugar problem

Let’s put some sugar into a glass and into a jug of water.
Is the water in the glass or the water in the jug sweeter?
If they are different, how do you make them the same sweet?

In the following figures a balance is used to measure the weights of the two empty containers, the weights of the containers full of water and finally their weights after adding the sugar


150g
750g
300g
1350g
zucchero
si mette
zucchero e
si mescola
zucchero
si mette
zucchero e
si mescola
320g
1390g

PRIMARY SCHOOL of PIETRAROJA (Benevento, Italy)

MATHEMATICS AND SCIENCE EXPERIMENTATION

Water and sugar

PROBLEM SITUATION PRESENTED
IN THE 1st / 2nd GRADE CLASS

Teachers: Michelina Venditto, Rosa Ferrara

School year 2009-2010

The pupils are presented with two similar bowls and are invited to observe them.

A lot of curiosity and interest rises immediately and the characteristics of the bowls and the quantity of water are soon noticed.

The pupils ask what needs to be done and the meaning of solution (water and sugar) is faced and discovered.

Everyone knew the concept of sweetness.

The teacher first invites them to turn around and, in the meantime, pour the different amounts of sugar into the water as established. Then invites them to mix to dissolve the sugar. Pupils have fun and take turns.

Teacher - **Which of the two solutions is sweeter?**

Manuela and Mariarosaria - In the yellow bowl there is more water and more sugar and therefore it is sweeter.

The others agree that we need to taste.

In turn, the first solution is tasted with a spoon.

Luciana - It is very sweet as a syrup.

Marco - It is very sweet.

The second solution is also tasted, that of the clear bowl.

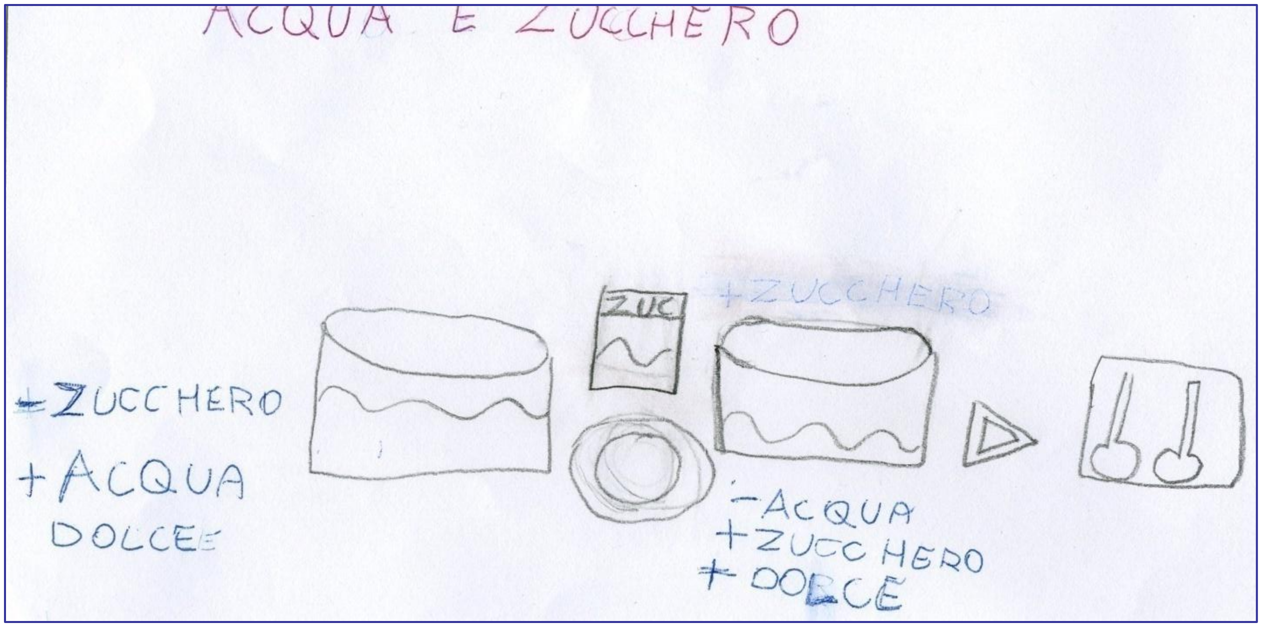
Manuela - It is sweeter because there is more sugar.

Luciana and Giancarlo - It is sweeter than before.

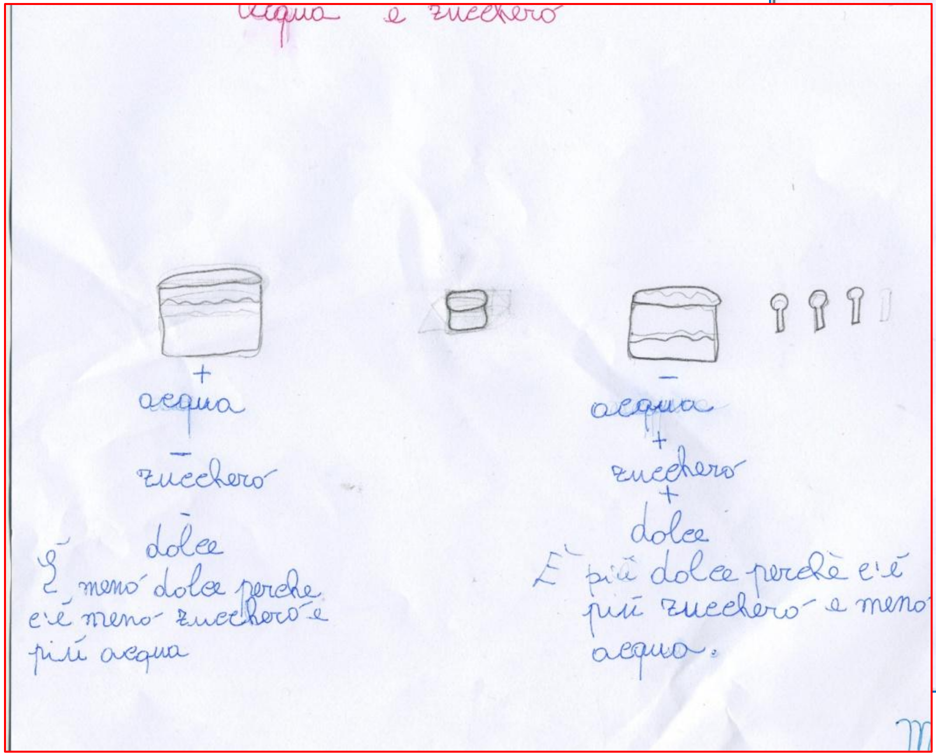
Marco **- It is sweeter because there is less water.**

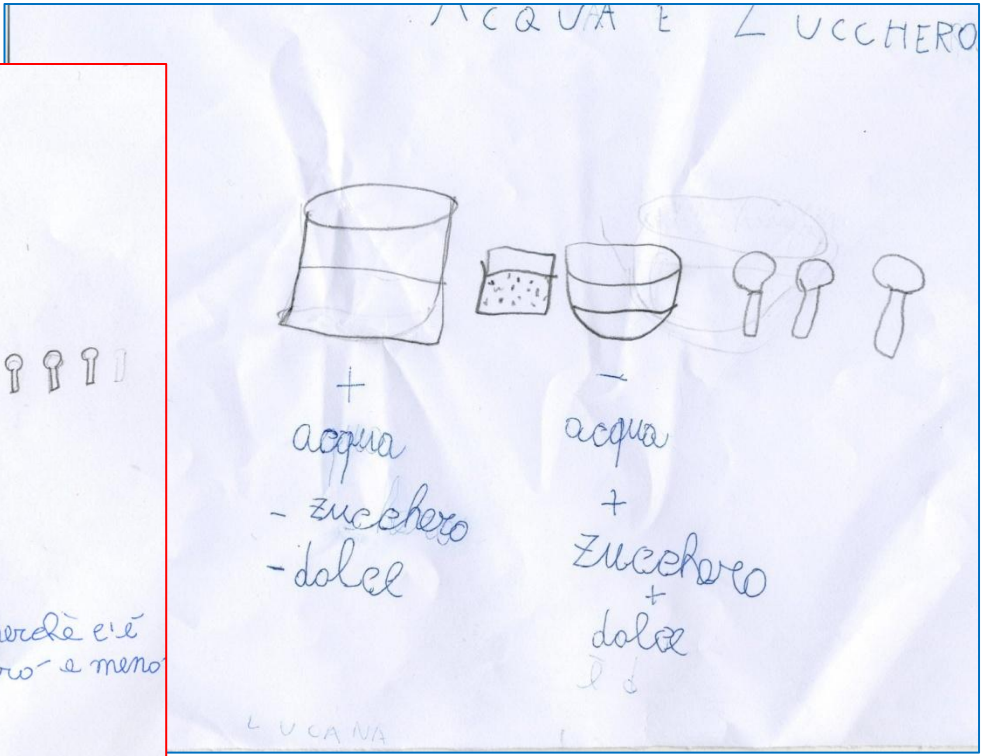
Rosanna - **There is more sugar and less water.**

At this point the children are invited to draw the observed situation: bowls, water, spoons, sugar, ... and to verbalize with simple words.



In the drawing, the pupils represented the experience they made and the objects they actually used.





The words used in the verbalization were **PLUS** and **MINUS** and when asked if they knew any symbols to indicate those same words they answered with the **arithmetic symbols of addition and subtraction** as the pupils had already used these symbols and therefore only transferred their acquaintances in the presented situation.

Teacher - If we wanted to make them the same sweet, what should we do?

All: Add + sugar.

Mariarosaria - I would add 3 tablespoons of sugar.

Teacher - So do you know how much sugar there is already?

All - No.

Teacher - Is it important to know?

Everyone - Yes.

The teacher reveals that in the large quantity of water there are 8 tablespoons of sugar and in the small one there are 4.

Rosanna - We need to see the amount of water and **divide** the water.

After a little discussion and observation, **we get to see how much water there is.**

Marco and Mariarosaria - The one with the bowl with more water.

Manuela - **We use glasses**.

Teacher - How many do I take?

Everyone - It takes a lot.

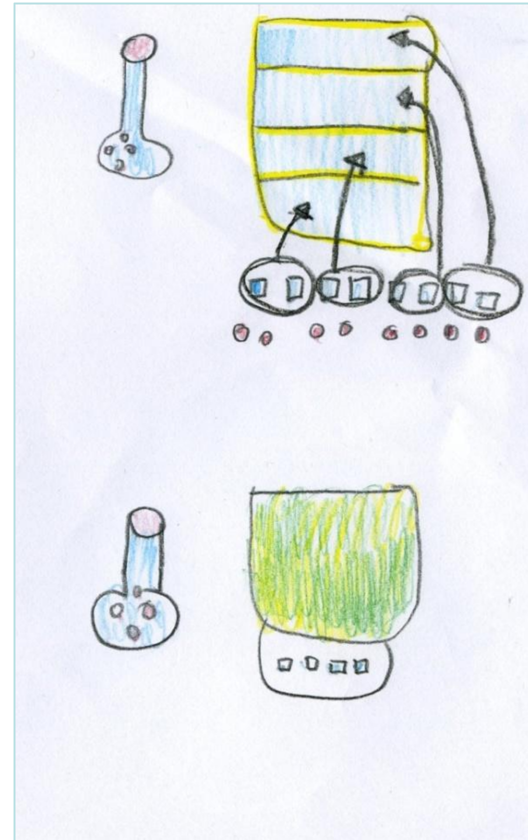
Teacher - Can we use something else?

Everyone - **The bottles**.

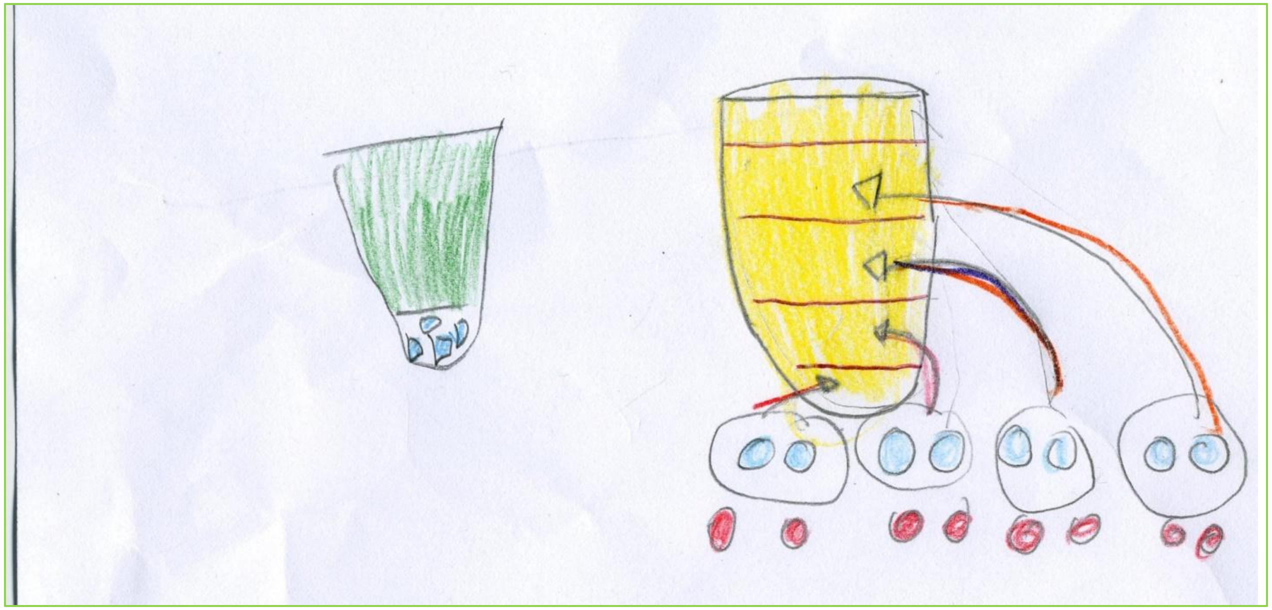
Less water is put into the large bottle and **the level is marked with a notch**. The same is put into the small bottle that fills up.

The same is done with more water, first in the large bottle and then in the small bottle and **it turns out that 4 bottles can be filled.**

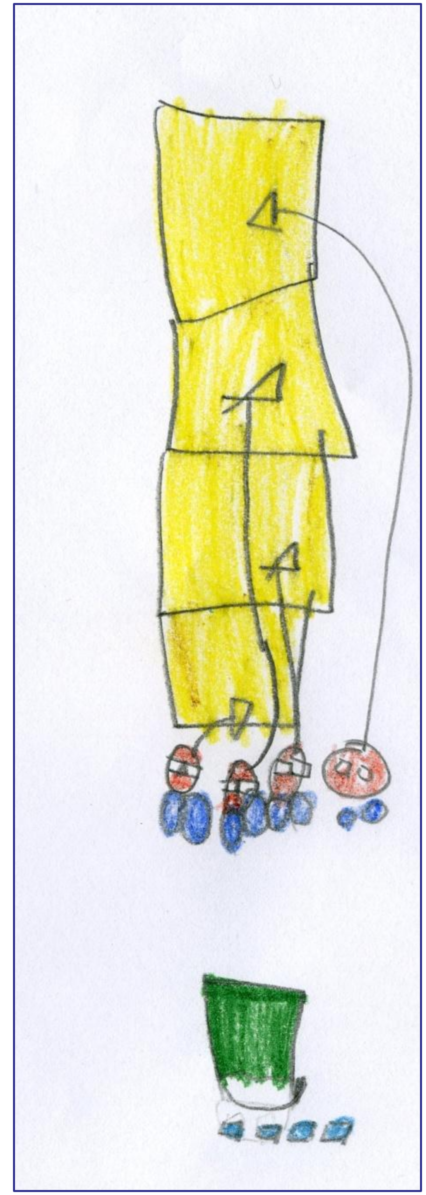
Each time a bottle is emptied, a notch is marked on the container. The eight spoons of sugar already present are symbolically represented with blue balls and are distributed equally for each notch.



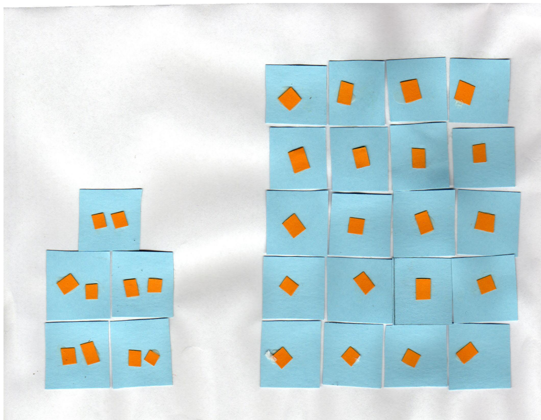
If in the small quantity of water there are 4 tablespoons of sugar (= one notch), in the large quantity there must be 4 tablespoons for each notch.



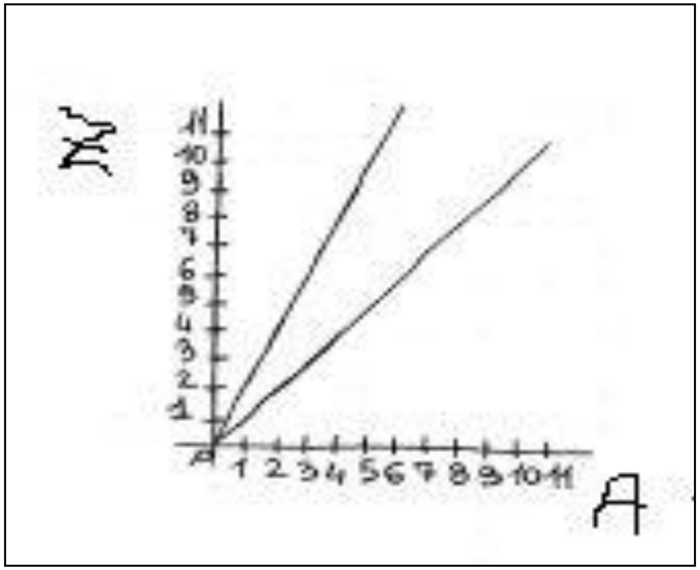
They proceed to add the spoons of sugar necessary to make the solutions the same sweet. This time the pupils choose a different color to indicate the amounts of sugar added.

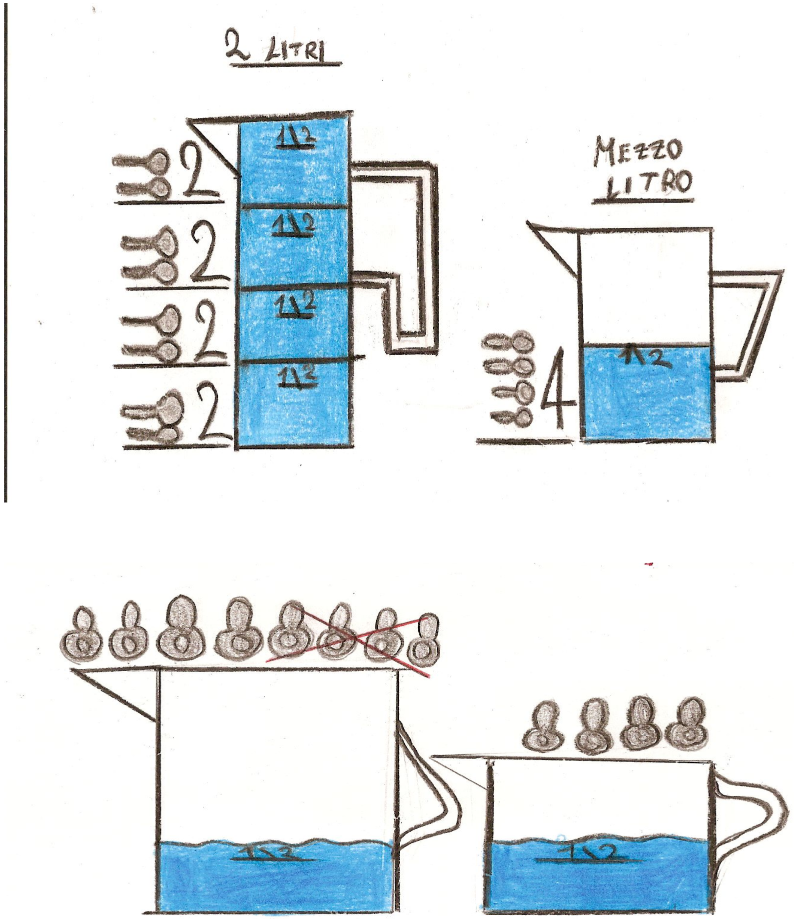


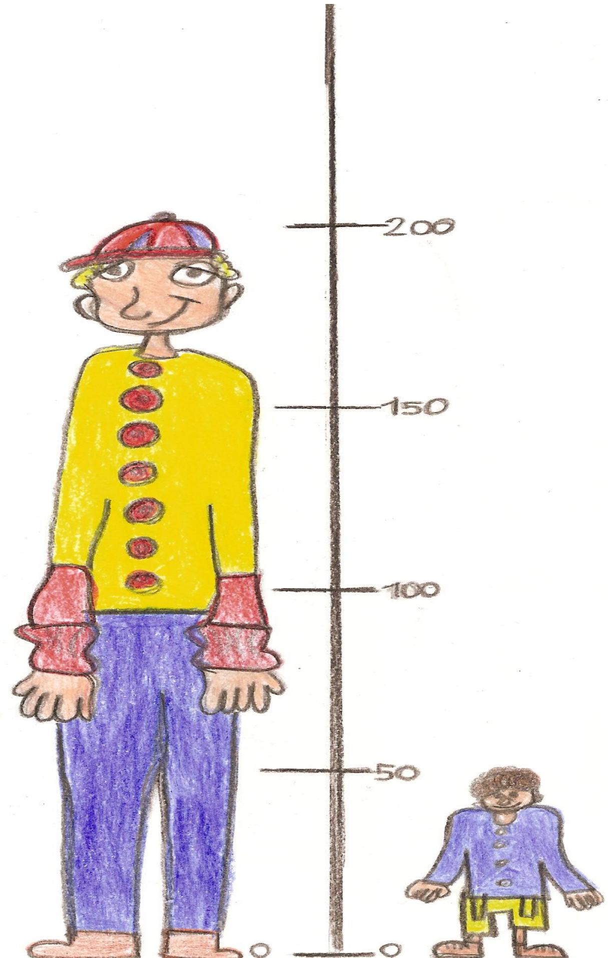
The amounts of sugar added were matched to the various notches with matching arrows.



	Pezzi di zucchero nel bicchiere	Pezzi di zucchero nella caraffa
1 pezzo d'acqua	2	1
2 pezzi d'acqua	4	2
3 pezzi d'acqua	6	3
4 pezzi d'acqua	8	4




2 LITRI
MEZZO
LITRO
2
2
2
2
4


200
150
100
50
0
0

As a conclusion

Putting together the considerations of the second and third points, the result is an idea of mathematics which I believe is very uncommon, that is a discipline which contains within it two apparently contradictory but in reality complementary instances:

1) the need for **rigor**
2) the ability to conceive **variation** and **change** as a foundation.

As a conclusion

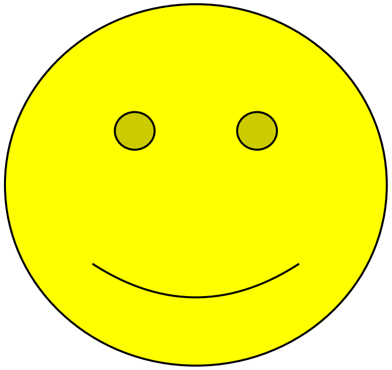
Transferring this conception of mathematics to the outside world should therefore serve:

- on the one hand, to get us used to the rigor in giving meaning to words, a rigor that is certainly not definitive but in constant construction;
- on the other hand, to consider that things change continuously and are all interdependent, and thus push us towards judgments that are never schematic and cutting but attentive to variations.

As a conclusion

By acquiring this idea, perhaps we could contribute both to attenuate the contrasts and at the same time to defeat or mitigate clear-cut and rigid conceptions, judgments and prejudices that divide men from one another. If at school we were really taught how mathematics can come into everyday life and into our daily thinking, it would be much more difficult for human beings to get to quarrel and clash.

Thank you
for your attention



grazie

Děkuji

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Additional slides

Verso la precisione terminologica. Elogio dell'ambiguità.

Ambiguità non vuol dire svalutare le caratteristiche della matematica né darla in pasto a chi nega il valore rigoroso della scienza e di questa disciplina in particolare.

Ambiguità significa riconoscere alle dette caratteristiche il ruolo di ideali cui costantemente tendere, e che appartengono ad una formalizzazione intesa come continuamente perfettibile, esito di lunghi processi storici e cognitivi.

Ma la matematica che si usa, e soprattutto quella che si apprende è decisamente un'altra. Può essere ampiamente modellata e personalizzata a seconda delle necessità e conosciuta e interpretata in modi diversi da persone diverse. Sta in ciò la difficoltà e insieme il fascino del compito di chi la insegna.

ELOGIO DELL'AMBIGUITA'.

L'ambiguità è una preziosa risorsa didattica.

Ambiguità non vuol dire svalutare le caratteristiche della matematica né darla in pasto a chi nega il valore rigoroso della scienza e di questa disciplina in particolare.

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E nel frattempo la matematica resta assoluta, ma solo quella, con buon pace di tutti, ufficiale o presunta tale.

Altri esempi dove l'uso colloquiale delle parole è (molto o poco) diverso da quello formale:

angolo
quadrato - rettangolo
cifra - numero
verticale - perpendicolare
possibile - probabile

asintoto
funzioni continue

Nomenclatura ambigua o infelice

- Numeri reali/immaginari
- Quoziente, dividere
- Altezza
- Positivo/negativo - affermazione/negazione
- La nomenclatura sui limiti
- Definizioni inclusive/esclusive in geometria
- Figura geometrica

Se non ci occupiamo serenamente di questi elementi di imperfezione presenti nella matematica della scuola,

corriamo seriamente il rischio di confinare la matematica in un dominio separato dalla realtà,

il rischio che la matematica si usi solo a scuola e poi nel resto della vita si regredisca ad approcci irrazionali e inadeguati.