Discrete wavelet transform, wavelets, and wavelet basis

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Lectures

1 Discrete wavelet transform, wavelets, and wavelet basis
2 Construction of spline wavelet basis
3 Wavelet methods for integro-differential equations
4 Wavelet methods for option pricing

Pdf files are available at https://kmd.fp.tul.cz/en/cb-profile/cerna

Outline

- Haar wavelet and Haar transform
- Wavelets and discrete wavelet transform
- Applications

Haar transform

Haar transform is the simplest type of the discrete wavelet transform (DWT). Haar wavelets were studied by A. Haar in 1910.

We focus on the wavelet analysis of

- vectors (discrete signals)
- matrices (images)
- real one-variable functions

First, we consider a vector $\mathbf{v}$ representing a discrete signal. Objective of DWT is to find a multiscale representation of $\mathbf{v}$ to analyze signal at different scales. Such a representation can be useful for sparse representation of a signal, compression, noise removal, detection of singularities, finding trend and periodicity, changes in variability, and so on.

Similarly, in the case of the fast Fourier transform (FFT) the signal is characterized by frequencies. An advantage of DWT is that it is local, while FFT is global.

Haar transform

filters: $\mathbf{h} = (h_1, h_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a scaling filter (low-pass filter)

$\mathbf{g} = (g_1, g_2) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is a wavelet filter (high-pass)

input vector: $\mathbf{v} = (v_1, \ldots v_n)$, n even

output vector: $\mathbf{c} = \left(c_1, \ldots c_{n/2}\right)$, $\mathbf{d} = \left(d_1, \ldots d_{n/2}\right)$

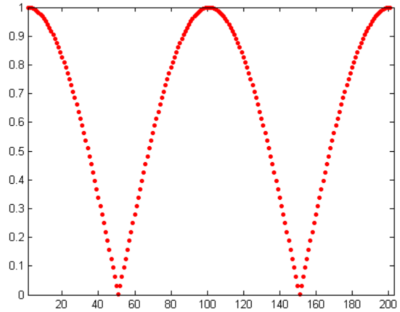
first step of dwt: $\mathbf{v} \to (\mathbf{c}, \mathbf{d})$

$$c_k = \sum_{i=1}^{2} h_i v_{2k-2+i},\ d_k = \sum_{i=1}^{2} g_i v_{2k-2+i},\ k = 1, \ldots, n/2$$

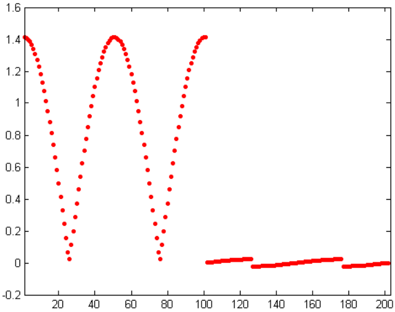
$$\begin{aligned}
c_1 &= h_1 v_1 + h_2 v_2 = \tfrac{v_1+v_2}{\sqrt{2}} & d_1 &= g_1 v_1 + g_2 v_2 = \tfrac{v_1-v_2}{\sqrt{2}} \\
c_2 &= h_1 v_3 + h_2 v_4 = \tfrac{v_3+v_4}{\sqrt{2}} & d_2 &= g_1 v_3 + g_2 v_4 = \tfrac{v_3-v_4}{\sqrt{2}} \\
c_3 &= h_1 v_5 + h_2 v_6 = \tfrac{v_5+v_6}{\sqrt{2}} & d_3 &= g_1 v_5 + g_2 v_6 = \tfrac{v_5-v_6}{\sqrt{2}} \\
&\vdots & &\vdots \\
c_{n/2} &= h_1 v_{n-1} + h_2 v_n = \tfrac{v_{n-1}+v_n}{\sqrt{2}} & d_{n/2} &= g_1 v_{n-1} + g_2 v_n = \tfrac{v_{n-1}-v_n}{\sqrt{2}}
\end{aligned}$$

Example 1. $v_i = |\cos(2\pi x_i)|$, $x_i = \frac{i-1}{199}$, $i = 1, \ldots, 200$

Input vector $\mathbf{v}$



Output vector $(\mathbf{c}, \mathbf{d})$



Haar transform: $\mathbf{v} \to \left[\mathbf{c}^M, \mathbf{d}^M, \ldots, \mathbf{d}^2, \mathbf{d}^1\right]$

$$\begin{array}{ccccccccccc}
\mathbf{v} & \xrightarrow{\mathbf{h}} & \mathbf{c}^1 & \xrightarrow{\mathbf{h}} & \mathbf{c}^2 & \xrightarrow{\mathbf{h}} & \ldots & \mathbf{c}^{M-1} & \xrightarrow{\mathbf{h}} & \mathbf{c}^M & . \\
 & \searrow^{\mathbf{g}} & & \searrow^{\mathbf{g}} & & \searrow^{\mathbf{g}} & & & \searrow^{\mathbf{g}} & & \\
 & \mathbf{d}^1 & & \mathbf{d}^2 & & \mathbf{d}^3 & \ldots & & \mathbf{d}^M & &
\end{array}$$

Example 2.

We consider a vector $\mathbf{v} = (9,\ 9,\ 8,\ 6,\ 7,\ 5,\ 6,\ 6)$ and we apply Haar transform on it.

First step:

$$\begin{aligned}(\mathbf{c}^1, \mathbf{d}^1) &= \left(\frac{9+9}{\sqrt{2}}, \frac{8+6}{\sqrt{2}}, \frac{7+5}{\sqrt{2}}, \frac{6+6}{\sqrt{2}}, \frac{9-9}{\sqrt{2}}, \frac{8-6}{\sqrt{2}}, \frac{7-5}{\sqrt{2}}, \frac{6-6}{\sqrt{2}}\right) \\ &= \left(9\sqrt{2},\ 7\sqrt{2},\ 6\sqrt{2},\ 6\sqrt{2},\ 0,\ \sqrt{2},\ \sqrt{2},\ 0\right)\end{aligned}$$

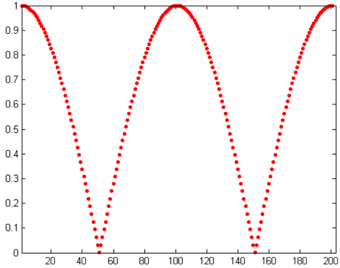
Second step:

$$\begin{aligned}(\mathbf{c}^2, \mathbf{d}^2, \mathbf{d}^1) &= \left(\frac{9\sqrt{2}+7\sqrt{2}}{\sqrt{2}}, \frac{6\sqrt{2}+6\sqrt{2}}{\sqrt{2}}, \frac{9\sqrt{2}-7\sqrt{2}}{\sqrt{2}}, \frac{6\sqrt{2}-6\sqrt{2}}{\sqrt{2}}, 0, \sqrt{2}, \sqrt{2}, 0\right) \\ &= \left(16,\ 12,\ 2,\ 0,\ 0,\ \sqrt{2},\ \sqrt{2},\ 0\right)\end{aligned}$$

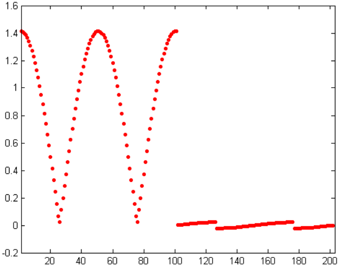
Third step:

$$\begin{aligned}(\mathbf{c}^3, \mathbf{d}^3, \mathbf{d}^2, \mathbf{d}^1) &= \left(\frac{16+12}{\sqrt{2}}, \frac{16-12}{\sqrt{2}}, 2,\ 0,\ 0,\ \sqrt{2},\ \sqrt{2},\ 0\right) \\ &= \left(14\sqrt{2},\ 2\sqrt{2},\ 2,\ 0,\ 0,\ \sqrt{2},\ \sqrt{2},\ 0\right)\end{aligned}$$

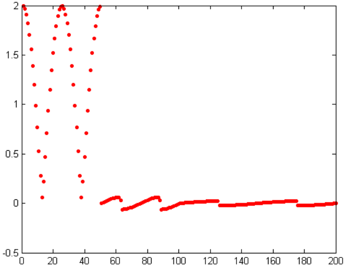
Example 1. Input vector **v**



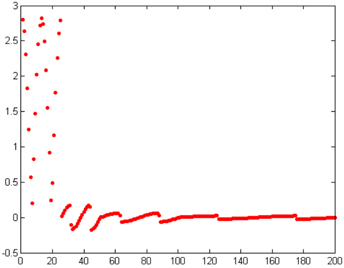
Transformed vector, $M = 1$



Transformed vector, $M = 2$



Transformed vector, $M = 3$



Inverse Haar transform

one step of inverse Haar transform: $(\mathbf{c}, \mathbf{d}) \to \mathbf{v}$

$$v_1 = h_1 c_1 + g_1 d_1 = \frac{c_1+d_1}{\sqrt{2}} \qquad v_2 = h_2 c_1 + g_2 d_1 = \frac{c_1-d_1}{\sqrt{2}}$$

$$v_3 = h_1 c_2 + g_1 d_2 = \frac{c_2+d_2}{\sqrt{2}} \qquad v_4 = h_2 c_2 + g_2 d_2 = \frac{c_2-d_2}{\sqrt{2}}$$

$$v_5 = h_1 c_3 + g_1 d_3 = \frac{c_3+d_3}{\sqrt{2}} \qquad v_6 = h_2 c_3 + g_2 d_3 = \frac{c_3-d_3}{\sqrt{2}}$$

$$\vdots \qquad\qquad \vdots$$

$$v_{n-1} = h_1 c_{n/2} + g_1 d_{n/2} = \frac{c_{n/2}+d_{n/2}}{\sqrt{2}} \qquad v_n = h_2 c_{n/2} + g_2 d_{n/2} = \frac{c_{n/2}-d_{n/2}}{\sqrt{2}}$$

Haar transform: $\left[\mathbf{c}^M, \mathbf{d}^M, \ldots, \mathbf{d}^2, \mathbf{d}^1\right] \to \mathbf{v}$

For $k = 1, \ldots, n/2^j$, $j = 0, \ldots, M-1$

$$c^j_{2k-1} = \frac{c^{j+1}_k + d^{j+1}_k}{\sqrt{2}}, \quad c^j_{2k} = \frac{c^{j+1}_k - d^{j+1}_k}{\sqrt{2}},$$

$\mathbf{v} := \mathbf{c}^0$

The inverse Haar transform can be visualized as the pyramid scheme

$$\begin{array}{ccccccccc}
\mathbf{c}^M & \xrightarrow{\mathbf{h}} & \mathbf{c}^{M-1} & \xrightarrow{\mathbf{h}} & \mathbf{c}^{M-2} & \longrightarrow & \dots & \mathbf{c}^1 & \xrightarrow{\mathbf{h}} & \mathbf{c}^0 \; . \\
 & \mathbf{g} \nearrow & & \mathbf{g} \nearrow & & \mathbf{g} \nearrow & & & \mathbf{g} \nearrow & \\
 & \mathbf{d}^M & & \mathbf{d}^{M-1} & & \mathbf{d}_{M-2} & \dots & & \mathbf{d}^1 &
\end{array}$$

Example 2.

We consider a transformed vector $\left(\mathbf{c}^3, \mathbf{d}^3, \mathbf{d}^2, \mathbf{d}^1\right) = \left(14\sqrt{2},\ 2\sqrt{2},\ 2,\ 0,\ 0,\ \sqrt{2},\ \sqrt{2},\ 0\right)$ and we want to reconstruct the original vector.

First step:

$$\begin{aligned}\left(\mathbf{c}^2, \mathbf{d}^2, \mathbf{d}^1\right) &= \left(\frac{14\sqrt{2}+2\sqrt{2}}{\sqrt{2}}, \frac{14\sqrt{2}-2\sqrt{2}}{\sqrt{2}}, 2\sqrt{2},\ 2,\ 0,\ 0,\ \sqrt{2},\ \sqrt{2},\ 0\right)\\ &= \left(16,\ 12,\ 2,\ 0,\ 0,\ \sqrt{2},\ \sqrt{2},\ 0\right)\end{aligned}$$

First step:

$$\begin{aligned}\left(\mathbf{c}^1, \mathbf{d}^1\right) &= \left(\frac{16+2}{\sqrt{2}}, \frac{16-2}{\sqrt{2}}, \frac{12+0}{\sqrt{2}}, \frac{12-0}{\sqrt{2}},\ 0,\ \sqrt{2},\ \sqrt{2},\ 0\right)\\ &= \left(9\sqrt{2},\ 7\sqrt{2},\ 6\sqrt{2},\ 6\sqrt{2},\ 0,\ \sqrt{2},\ \sqrt{2},\ 0\right)\end{aligned}$$

Third step:

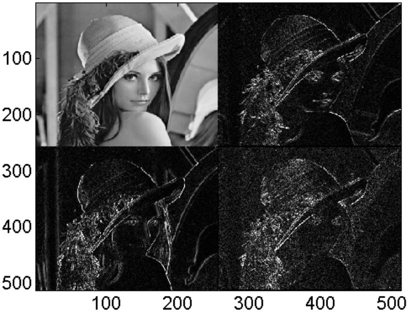
$$\begin{aligned}\mathbf{c}^0 &= \left(\frac{9\sqrt{2}+0}{\sqrt{2}}, \frac{9\sqrt{2}-0}{\sqrt{2}}, \frac{7\sqrt{2}+\sqrt{2}}{\sqrt{2}}, \frac{7\sqrt{2}-\sqrt{2}}{\sqrt{2}}, \frac{6\sqrt{2}+\sqrt{2}}{\sqrt{2}}, \frac{6\sqrt{2}-\sqrt{2}}{\sqrt{2}}, \dots\right)\\ &= \left(9,\ 9,\ 8,\ 6,\ 7,\ 5,\ 6,\ 6\right)\end{aligned}$$

2D Haar transform

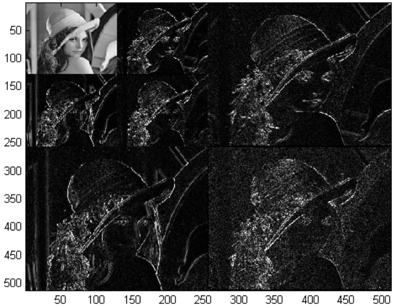
Grayscale image is represented by a matrix. Each pixel of the image is represented by one element of the matrix, the value of the element is of type uint8 (uint16, ...) and characterizes the shade of gray (0 - black, 255 - white). 2D DWT of the matrix is obtained by applying DWT on the rows and on the columns.

Original image

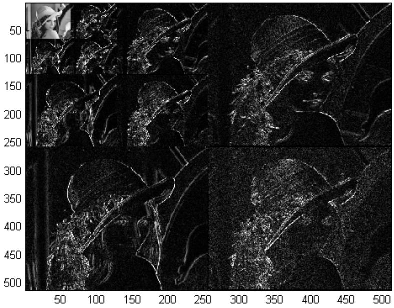
Transformed image



Transformed image, $M = 2$



Transformed image, $M = 3$



Haar wavelets

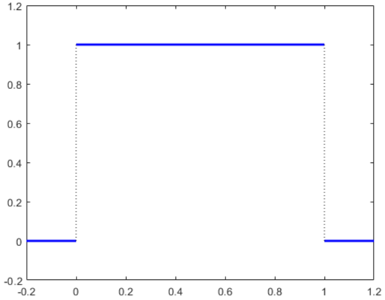
Let f be a function defined on the interval $[0, 1]$. Our objective is to find a multiscale representation (or approximation) of the function f, i.e. to characterize a function f using functions on certain scales (levels).

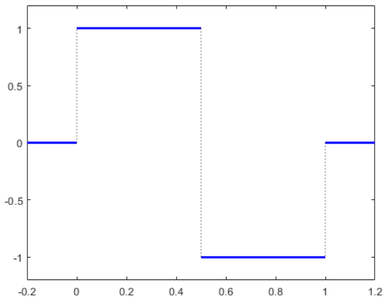
Such representation is useful for sparse representation of f, sparse representation of operators, adaptive solution of PDEs, integral equations, PIDEs, preconditioning of systems resulting from discretization of PDEs, etc.

In the case of approximation by Haar wavelets, we approximate f by a piecewise constant function.

Haar scaling function | **Haar wavelet**

$$\phi(x) = \chi_{[0,1)}(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & x \notin [0,1) \end{cases}, \quad \psi(x) = \begin{cases} 1 & x \in [0,1/2) \\ -1 & x \in [1/2,1) \\ 0 & x \notin [0,1) \end{cases}$$





We denote by $L^2(0,1)$ the space of all real valued functions defined on $(0,1)$ such that

$$\int_0^1 f^2(x)\,dx < \infty.$$

Haar wavelet basis of the space $L^2(0,1)$ is generated using translations and dilations of ϕ and ψ.

For $j,k \in \mathbb{Z}$ we define

$$\phi_{j,k}(x) = 2^{j/2}\phi\left(2^j x - k\right), \quad \psi_{j,k}(x) = 2^{j/2}\psi\left(2^j x - k\right).$$

The parameter j is called a scale or a level and the parameter k is a translation factor.

Theorem

The set

$$\Psi = \left\{ \phi, \psi_{j,k}, j \geq 0, k = 0, \ldots, 2^j - 1 \right\}$$

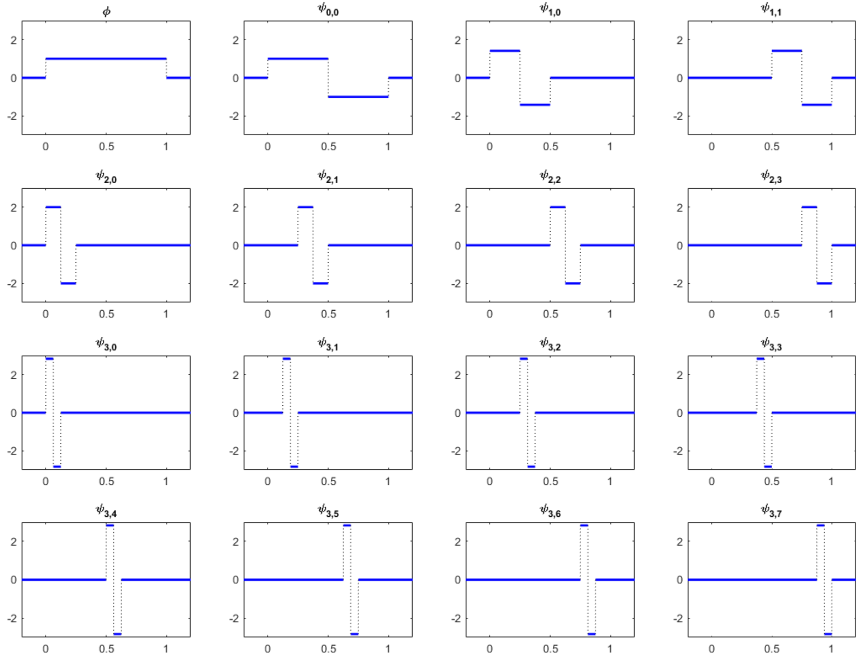
is an orthonormal basis of the space $L^2(0,1)$*,*
i.e. Ψ *generates* $L^2(0,1)$*,*

$$\int_{\mathbb{R}} \phi(x)\,\phi(x)\,dx = 1, \quad \int_{\mathbb{R}} \phi(x)\,\psi_{j,k}(x)\,dx = 0,$$

and

$$\int_{\mathbb{R}} \psi_{m,l}(x)\,\psi_{j,k}(x)\,dx = \begin{cases} 1 & j = m, k = l, \\ 0 & \text{otherwise}. \end{cases}$$

The set Ψ is called a Haar wavelet basis.



ϕ
$\psi_{0,0}$
$\psi_{1,0}$
$\psi_{1,1}$
$\psi_{2,0}$
$\psi_{2,1}$
$\psi_{2,2}$
$\psi_{2,3}$
$\psi_{3,0}$
$\psi_{3,1}$
$\psi_{3,2}$
$\psi_{3,3}$
$\psi_{3,4}$
$\psi_{3,5}$
$\psi_{3,6}$
$\psi_{3,7}$

Approximation of functions

Corollary. Any $f \in L^2(0,1)$ has a representation

$$f = c\phi + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k},$$

where

$$c = \int_0^1 f(x)\,\phi(x)\,dx, \quad d_{j,k} = \int_0^1 f(x)\,\psi_{j,k}(x)\,dx.$$

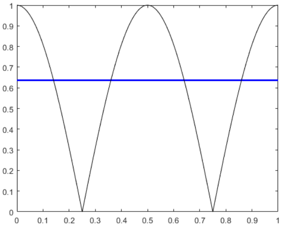
Hence, the function f can be approximated by a function

$$f_J = c\phi + \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}.$$

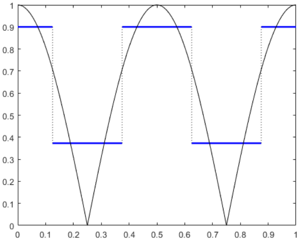
Example 3. $f(x) = |\cos(2\pi x)|$, $x \in [0,1]$

$$
\begin{aligned}
c &= \int_0^1 f(x)\,\phi(x)\,dx = \int_0^1 |\cos(2\pi x)|\,1dx = \frac{2}{\pi} \\
d_{0,0} &= \int_0^1 f(x)\,\psi_{0,0}(x)\,dx \\
&= \int_0^{1/2} |\cos(2\pi x)|\,dx - \int_{1/2}^1 |\cos(2\pi x)|\,dx = 0 \\
f_0 &= c\phi + d_{0,0}\psi_{0,0} = \frac{2}{\pi}
\end{aligned}
$$

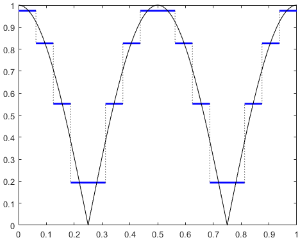
Approximation f_0



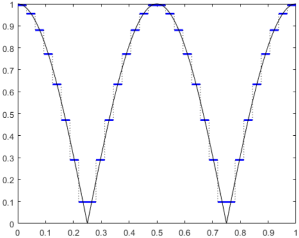
Approximation f_2



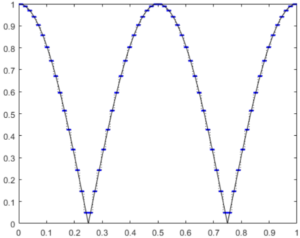
Approximation f_3



Approximation f_4



Approximation f_5



Relation between scaling function, wavelet and scaling and wavelet filters

The scaling function ϕ and the scaling filter $\mathbf{h}$ satisfy a scaling equation

$$\phi(x) = h_1\phi_{1,0}(x) + h_2\phi_{1,1}(x) = h_1\sqrt{2}\phi(2x) + h_2\sqrt{2}\phi(2x-1)$$

and ϕ, ψ, and $\mathbf{g}$ are interrelated by a wavelet equation

$$\psi(x) = g_1\phi_{1,0}(x) + g_2\phi_{1,1}(x) = g_1\sqrt{2}\phi(2x) + g_2\sqrt{2}\phi(2x-1).$$

Vanishing moments

Haar wavelet has one vanishing moment, i.e. $\int_0^1 \psi(x)\,dx = 0$.
Therefore, all wavelets $\psi_{j,k}$ have vanishing moments, i.e. $\int_0^1 \psi_{j,k}(x)\,dx = 0$.

Theorem. The coefficients $d_{j,k}$ satisfy $|d_{j,k}| \leq C2^{-3j/2}$ under the assumption that f has continuous derivative on $\left[k/2^j, (k+1)/2^j\right]$.

Wavelets

We use the standard notation $L^2(\mathbb{R})$ for the space of real valued functions defined on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} f^2(x)\, dx < \infty.$$

The inner product in this space is defined by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\, g(x)\, dx$$

and the norm is defined by

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

The symbol $C^m(a, b)$ denotes the space of m-times continuously differentiable functions on (a, b).

Definition: The sequence of spaces $V_j \subset L^2(\mathbb{R})$, $j \geq j_0$, is called a multiresolution analysis, if $V_j \subset V_{j+1}$, the closure of the span of $\cup_{j \geq j_0} V_j$ is $L^2(\mathbb{R})$, and if there exists a function ϕ such that $\Phi_j = \{\phi_{j,k}, k \in \mathbb{Z}\}$, where

$$\phi_{j,k}(x) = 2^{j/2}\phi\left(2^j x - k\right),$$

is a basis of V_j. The function ϕ is called a scaling function.

Let us assume that the support of ϕ is $[0, M-1]$. Since $V_j \subset V_{j+1}$ and Φ_j is a basis of V_j, there exists a vector $\mathbf{h} = (h_1, \ldots, h_M)$ such that

$$\phi(x) = \sum_{k=1}^{M} h_k \phi_{1,k-1}(x) = \sum_{k=1}^{M} h_k \sqrt{2}\phi(2x + 1 - k).$$

This equation is called a scaling equation and $\mathbf{h}$ is called a scaling filter (or low-pass filter).

Let W_j be such that $V_j \oplus W_j = V_{j+1}$ and let ψ be a function such that

$$\Psi_j = \{\psi_{j,k}, k \in \mathbb{Z}\}$$

is a basis of W_j.

Since $W_j \subset V_{j+1}$, there exists coefficients $\mathbf{g} = (g_1, \ldots, g_N)$ such that

$$\psi(x) = \sum_{k=1}^{N} g_k \phi_{1,k-1}(x) = \sum_{k=1}^{N} g_k \sqrt{2} \phi(2x + 1 - k).$$

This equation is called a wavelet equation and $\mathbf{g}$ is called a wavelet filter (high pass filter).

Definition: Under the above notation a family

$$\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0}^{\infty} \Psi_j$$

is called a wavelet basis of the space $L^2(\mathbb{R})$, if Ψ is a Riesz basis of the space $L^2(\mathbb{R})$, i.e. there exist constants $0 < c < C < \infty$ such that

$$\begin{aligned} c\left(\sum_{k\in\mathbb{Z}} c_{j_0,k}^2 + \sum_{j=j_0}^{\infty}\sum_{k\in\mathbb{Z}} d_{j,k}^2\right) &\leq \int_{\mathbb{R}}\left(\sum_{k\in\mathbb{Z}} c_{j_0,k}\phi_{j_0,k} + \sum_{j=j_0}^{\infty}\sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}\right)^2 dx \\ &\leq C\left(\sum_{k\in\mathbb{Z}} c_{j_0,k}^2 + \sum_{j=j_0}^{\infty}\sum_{k\in\mathbb{Z}} d_{j,k}^2\right). \end{aligned}$$

The function ψ is called a wavelet. If Ψ is an orthonormal basis, the function ψ is called an orthonormal wavelet.

Theorem. To any Riesz basis Ψ there exists a biorthogonal Riesz basis $\tilde{\Psi}$ and $\tilde{\Psi}$ has the similar structure as Ψ, i.e there exist functions $\tilde{\phi}$ and $\tilde{\psi}$ such that

$$\tilde{\Psi} = \left\{ \tilde{\phi}_{j_0,k}, \tilde{\psi}_{j,k}, j \geq j_0, k \in \mathbb{Z} \right\},$$

where

$$\tilde{\phi}_{j,k}(x) = 2^{j/2} \tilde{\phi}\left(2^j x - k\right), \quad \tilde{\psi}_{j,k}(x) = 2^{j/2} \tilde{\psi}\left(2^j x - k\right).$$

Biorthogonality means that

$$\begin{aligned} \left\langle \phi_{j,k}, \tilde{\phi}_{m,l} \right\rangle &= \delta_{j,m}\delta_{k,l}, \quad \left\langle \phi_{j,k}, \tilde{\psi}_{m,l} \right\rangle = 0, \\ \left\langle \psi_{j,k}, \tilde{\phi}_{m,l} \right\rangle &= 0, \qquad\quad \left\langle \psi_{j,k}, \tilde{\psi}_{m,l} \right\rangle = \delta_{j,m}\delta_{k,l}. \end{aligned}$$

The function $\tilde{\phi}$ is called a dual scaling function and the function $\tilde{\psi}$ is called a dual wavelet. Dual scaling filter $\tilde{\mathbf{h}}$ and dual wavelet filter $\tilde{\mathbf{g}}$ are defined similarly as $\mathbf{h}$ and $\mathbf{g}$.

Corollary. Any $f \in L^2(\mathbb{R})$ has a representation

$$f(x) = \sum_{k \in \mathbb{Z}} c_{j_0,k} \phi_{j_0,k} + \sum_{j=j_0}^{\infty} \sum_{k \in \mathbb{Z}} d_{j,k} \psi_{j,k},$$

where

$$c_{j,k} = \left\langle f, \tilde{\phi}_{j,k} \right\rangle, \quad d_{j,k} = \left\langle f, \tilde{\psi}_{j,k} \right\rangle.$$

The function f can be approximated by a function

$$f_J(x) = \sum_{k \in \mathbb{Z}} c_{j_0,k} \phi_{j_0,k} + \sum_{j=j_0}^{J} \sum_{k \in \mathbb{Z}} d_{j,k} \psi_{j,k}.$$

Vanishing moments

We say that a wavelet ψ has L vanishing moments, if

$$\int_{\mathbb{R}} x^k \psi(x)\, dx = 0, \quad k = 0, \dots, L-1.$$

Theorem: If $\tilde{\psi}$ has L vanishing moments and $f \in C^L(0,1)$, then

$$\|f - f_J\| \leq C 2^{-LJ}.$$

Theorem: If $\tilde{\psi}$ has L vanishing moments and $f \in C^L\left(supp\, \tilde{\psi}_{j,k}\right)$, then

$$|d_{j,k}| \leq C 2^{-j(L+1/2)}.$$

Discrete wavelet transform

filters: $\mathbf{h} = (h_1, h_2, \dots h_M)$ - scaling (low-pass) filter,
$\mathbf{g} = (g_1, g_2, \dots g_N)$ - wavelet (high-pass) filter

one step of DWT: $\mathbf{v} \to (\mathbf{c}, \mathbf{d})$

$$c_k = \sum_{i=1}^{M} h_i v_{2k-2+i}, \; d_k = \sum_{i=1}^{N} g_i v_{2k-2+i}, \; k \in \mathbb{Z}$$

(If necessary, $\mathbf{v}$ can be extended, e.g. by zero.)

DWT $\mathbf{v} \to \left[\mathbf{c}^M, \mathbf{d}^M, \dots, \mathbf{d}^2, \mathbf{d}^1\right]$

$$\begin{array}{ccccccccccc}
\mathbf{v} & \xrightarrow{\mathbf{h}} & \mathbf{c}^1 & \xrightarrow{\mathbf{h}} & \mathbf{c}^2 & \xrightarrow{\mathbf{h}} & \dots & \mathbf{c}^{M-1} & \xrightarrow{\mathbf{h}} & \mathbf{c}^M & . \\
 & \searrow^{\mathbf{g}} & & \searrow^{\mathbf{g}} & & \searrow^{\mathbf{g}} & & & \searrow^{\mathbf{g}} & & \\
 & \mathbf{d}^1 & & \mathbf{d}^2 & & \mathbf{d}^3 & \dots & & \mathbf{d}^M & &
\end{array}$$

Inverse discrete wavelet transform (IDWT)

The inverse transform uses biorthogonal filters $\tilde{\mathbf{h}}$ and $\tilde{\mathbf{g}}$ and is given by the formula:

$$c_k^j = \sum_{n\in\mathbb{Z}} \tilde{h}_{k-2n} c_{n+1}^{j+1} + \sum_{n\in\mathbb{Z}} \tilde{g}_{k-2n} d_{n+1}^{j+1},$$

where $j = M-1, \ldots, 0$ and $k \in \mathbb{Z}$.

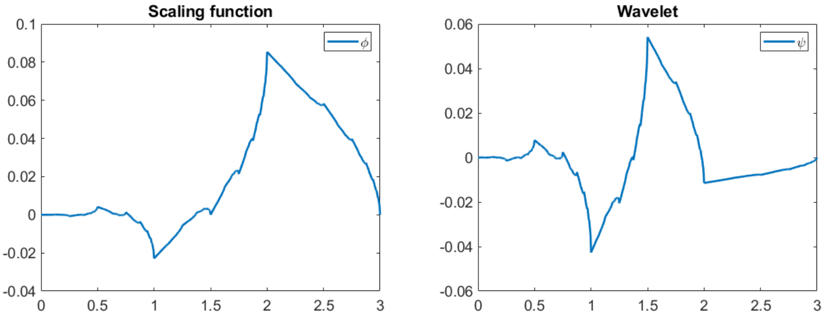
Schematically, IDWT can be visualized as the pyramid scheme

$$\begin{array}{ccccccccccc}
\mathbf{c}^M & \overset{\tilde{\mathbf{h}}}{\longrightarrow} & \mathbf{c}^{M-1} & \overset{\tilde{\mathbf{h}}}{\longrightarrow} & \mathbf{c}^{M-2} & \longrightarrow & \ldots & \mathbf{c}^1 & \overset{\tilde{\mathbf{h}}}{\longrightarrow} & \mathbf{c}^0 & . \\
 & \tilde{\mathbf{g}} \nearrow & & \tilde{\mathbf{g}} \nearrow & & \tilde{\mathbf{g}} \nearrow & & & \tilde{\mathbf{g}} \nearrow & & \\
 & \mathbf{d}^M & & \mathbf{d}^{M-1} & & \mathbf{d}_{M-2} & \ldots & & \mathbf{d}^1 & &
\end{array}$$

Example 4. Daubechies Db2 filters are given by

$$\mathbf{h} = \frac{1}{\sqrt{2}}\left(\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8}\right),$$
$$\mathbf{g} = \frac{1}{\sqrt{2}}\left(\frac{1-\sqrt{3}}{8}, \frac{-3+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{-1-\sqrt{3}}{8}\right)$$

Daubechies Db2 wavelet is an orthogonal wavelet, i.e. $\tilde{\mathbf{h}} = \mathbf{h}$, $\tilde{\mathbf{g}} = \mathbf{g}$, $\tilde{\phi} = \phi$, $\tilde{\psi} = \psi$, and has two vanishing moments. The scaling function is given as a solution of the scaling equation, both the scaling function and wavelet have no analytic expressions, they have not derivative.



Example 4.

We consider a vector $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5, v_6)$.

We apply one step of DWT and we obtain a vector $(\mathbf{c}, \mathbf{d}) = (c_0, c_1, c_2, c_3, d_0, d_1, d_2, d_3)$, where

$$\begin{aligned}
c_0 &= h_1 0 + h_2 0 + h_3 v_1 + h_4 v_2 \\
c_1 &= h_1 v_1 + h_2 v_2 + h_3 v_3 + h_4 v_4 \\
c_2 &= h_1 v_3 + h_2 v_4 + h_3 v_5 + h_4 v_6 \\
c_3 &= h_1 v_5 + h_2 v_6 + h_3 0 + h_4 0 \\
d_0 &= g_1 0 + g_2 0 + g_3 v_1 + g_4 v_2 \\
d_1 &= g_1 v_1 + g_2 v_2 + g_3 v_3 + g_4 v_4 \\
d_2 &= g_1 v_3 + g_2 v_4 + g_3 v_5 + g_4 v_6 \\
d_3 &= g_1 v_5 + g_2 v_6 + g_3 0 + g_4 0
\end{aligned}$$

Extension of a vector **v**

- zero padding (zpd) -
  ... 0 0 0 0 **1 2 3 4 5** 0 0 0 0 ...
- symmetrization (sym) -
  ... 4 3 2 1 **1 2 3 4 5** 5 4 3 2 ... (half point),
  ... 5 4 3 2 **1 2 3 4 5** 4 3 2 1 ... (whole point)
- asymmetric padding (asym) -
  ... -4 -3 -2 -1 **1 2 3 4 5** -5 -4 -3 -2 ...
- smooth padding of order 1 (sp1) -
  ... -3 -2 -1 0 **1 2 3 4 5** 6 7 8 9 ...
- smooth padding of order 0 (sp0) -
  ... 1 1 1 1 **1 2 3 4 5** 5 5 5 5 ...
- periodic padding (ppd) -
  ... 2 3 4 5 **1 2 3 4 5** 1 2 3 4 ...

Applications of DWT

- sparse representation, compression, dimensionality reduction of signals, data, and images
- detection of changes, singularities, edges
- time series analysis (to determine trend, periodicity, changes in variability, prediction)
- removing noise, smoothing
- numerical solution of operator equations (to transform a discretization matrix for a scaling basis to a discretization matrix for a wavelet basis)

Applications of wavelet approximation

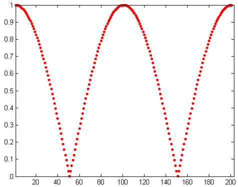
- sparse representation, compression of functions
- removing noise, smoothing, detection of changes of variability for continuous signals
- sparse representation and compression of operators
- adaptive solution of PDEs, integral equations and PIDEs
- preconditioning of large systems arising from discretization of operator equations
- numerical solution of high-dimensional problems overcoming the curse of dimensionality
- high-order approximation of functions

Example 1. Sparse approximation

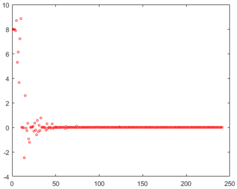
$v_i = |\cos(2\pi x_i)|$, $x_i = \frac{i-1}{199}$, $i = 1, \ldots, 200$

We apply DWT using Db2 wavelet family, $M = 6$ levels, and symmetric extension (halfpoint) of the vector $\mathbf{v}$. To obtain a sparse approximation we apply thresholding with threshold $T = 0.03$. The error of approximation is characterized by $e = \max |v_i - w_i|$, where $\mathbf{w}$ is a reconstructed vector.

Input vector $\mathbf{v}$



Transformed vector



Error

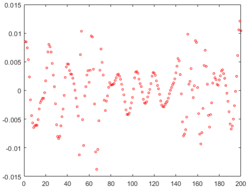


Table: Comparison of sparse approximations for several wavelet filters, nnz is the number of nonzero elements

	Haar		Db2		Db3	
T	nnz	error	nnz	error	nnz	error
0.1	40	0.0564	23	0.0598	26	0.0459
0.05	65	0.0350	35	0.0290	32	0.0395
0.03	79	0.0237	38	0.0239	38	0.0176

Table: Comparison of sparse approximations for several signal extensions, nnz is the number of nonzero elements, $T = 0.03$.

	Haar		Db2		Db3	
mode	*nnz*	error	*nnz*	error	*nnz*	error
sym	79	0.0237	38	0.0239	38	0.0176
zpd	82	0.0237	49	0.0239	57	0.0174
asym	79	0.0237	50	0.0239	64	0.0174
sp1	82	0.0237	33	0.0239	34	0.0176
sp0	79	0.0237	35	0.0239	35	0.0176
ppd	81	0.0237	38	0.0239	40	0.0176

Example 5: Image of Lena

We compute the decompositions for several image extensions.

We decompose the image on five levels using biorthogonal spline wavelets 3/5. Then we threshold the wavelet coefficients greater than 100 and we reconstruct an image. Let I and $\hat{I}$ be arrays of the size 512×512 characterizing grey levels in the original image and the reconstructed image, respectively. We compute

$$\mathrm{K} := \frac{\text{number of nonzero coefficients}}{\text{number of pixels in an original image}}$$

and

$$\text{relative error} := \sqrt{\frac{\sum_{i,j=1}^{512} \left(I(i,j) - \hat{I}(i,j) \right)^2}{\sum_{i,j=1}^{512} I(i,j)^2}}.$$

Furthermore, we compute the boundary error, i.e. the relative error for the area near the boundary.

	Image of Lena		
method	K	error	boundary error
sp0	0.0210	0.0631	0.0457
sp1	0.0228	0.0631	0.0459
sym	0.0225	0.0632	0.0469
ppd	0.0293	0.0669	0.1226
asym	0.0369	0.0690	0.1510
zpd	0.0201	0.0735	0.1969

Table: Errors and ratios K for several image compression methods.

References

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