Construction of spline wavelet bases

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Lectures

1. Discrete wavelet transform, wavelets, and wavelet basis
2. Construction of spline wavelet bases
3. Wavelet methods for integro-differential equations
4. Wavelet methods for option pricing

Pdf files are available at https://kmd.fp.tul.cz/en/cb-profile/cerna

Outline

- B-splines
- Construction of spline wavelet bases on the real line
- Wavelet bases on the interval
- Wavelet bases on product domains
- Construction of quadratic-spline wavelet bases

B-splines

Definition: The (cardinal) B-spline B_N of degree N, $N \in \mathbb{N}$, is defined by $B_1 = \chi_{[0,1)}$ and

$$B_N(x) = B_1 * B_{N-1}(x) = \int_{\mathbb{R}} B_1(t) B_{N-1}(x - t)\, dt, \quad N \geq 2.$$

Theorem. For $N \in \mathbb{N}$ the functions B_N have the following properties:

1) B_N is supported in $[0, N]$.

2) $B_N(x) > 0$ for all $x \in (0, N)$.

3) The function B_N is symmetric with respect to the point $\frac{N}{2}$, i.e.

$$B_N\left(\frac{N}{2} - x\right) = B_N\left(\frac{N}{2} + x\right) \quad \text{for all} \quad x \in \mathbb{R}.$$

4) $\int_{\mathbb{R}} B_N(x)\, dx = 1$.

5) For all $x \in \mathbb{R}$ we have

$$B_N(x) = \frac{1}{(N-1)!} \sum_{k=0}^{N} (-1)^k \binom{N}{k} (x-k)_+^{N-1}, \; x_+^N = (\max\{0, x\})^N.$$

6) B_N generates the multiresolution spaces

$$V_j = \left\{ f \in L^2(\mathbb{R}) \cap C^{N-1}(\mathbb{R}) : f|_{\left(\frac{k}{2^j}, \frac{k+1}{2^j}\right)} \text{ is a polynomial of order } \le N, k \in \mathbb{Z} \right\}.$$
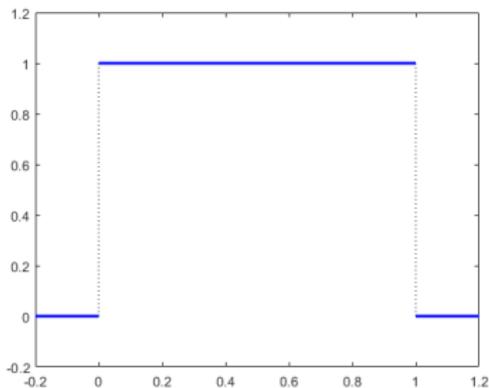
7) B_N satisfies a scaling equation
$B_N(x) = \sum_{n \in \mathbb{Z}} h_n \sqrt{2} B_N(2x - n)$ with scaling coefficients

$$h_n = 2^{-N+1/2} \binom{N}{n} \quad \text{for} \quad n = 0, \dots N, \quad h_n = 0 \quad \text{otherwise.}$$

Example 1. B-spline of order $N = 1$

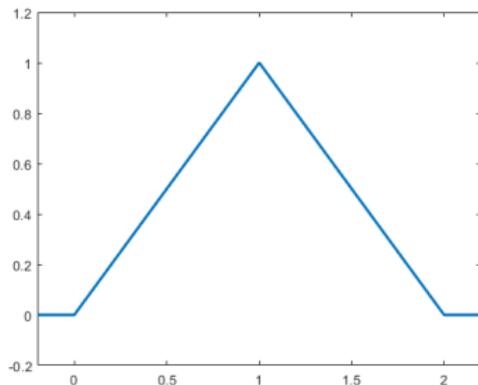B-spline of order $N = 1$ is a Haar scaling function

$$B_1(x) = \phi(x) = \chi_{[0,1)}(x) = \begin{cases} 1 & x \in [0,1), \\ 0 & x \notin [0,1). \end{cases}$$

Example 2. B-spline of order $N = 2$

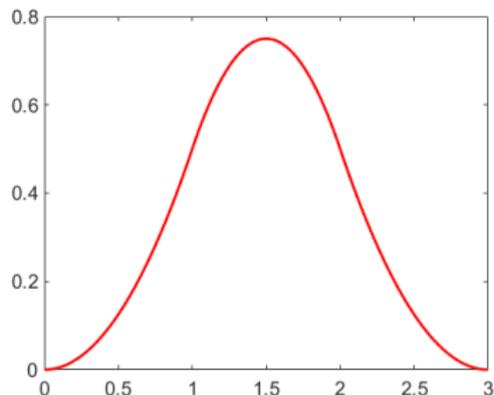B-spline of order $N = 2$ is a linear B-spline called also a hat function.

$$B_2 (x) = \int_{\mathbb{R}} B_1 (t) B_1 (x - t) \, dt = \begin{cases} x & x \in [0, 1), \\ 2 - x & x \in [1, 2), \\ 0 & \text{otherwise}. \end{cases}$$

Example 3. B-spline of order $N = 3$

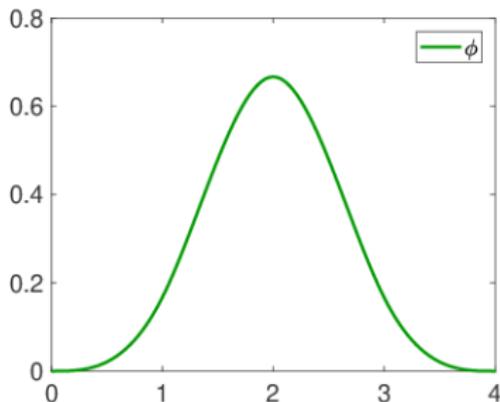B-spline of order $N = 3$ is a quadratic B-spline.

$$B_3(x) = \int_{\mathbb{R}} B_1(t) B_2(x - t) \, dt = \begin{cases} \frac{x^2}{2}, & x \in [0, 1], \\ -x^2 + 3x - \frac{3}{2}, & x \in [1, 2], \\ \frac{x^2}{2} - 3x + \frac{9}{2}, & x \in [2, 3], \\ 0, & \text{otherwise.} \end{cases}$$

Example 3. B-spline of order $N = 4$

B-spline of order $N = 4$ is a cubic B-spline.

$$B_4(x) = \int_{\mathbb{R}} B_1(t) B_3(x - t) \, dt = \begin{cases} \frac{x^3}{6}, & x \in [0, 1], \\ -\frac{x^3}{2} + 2x^2 - 2x + \frac{2}{3}, & x \in [1, 2], \\ \frac{x^3}{2} - 4x^2 + 10x - \frac{22}{3}, & x \in [2, 3], \\ \frac{(4-x)^3}{6}, & x \in [3, 4], \\ 0, & \text{otherwise.} \end{cases}$$

Spline wavelet bases on the real line

The following construction was proposed in [Cohen, Daubechies, and Feauveau, 1992].

We define a primal scaling function as $\phi = B_N$ for a chosen $N \in \mathbb{N}$. We choose the number of vanishing moments of a wavelet ψ as $\tilde{N} \in \mathbb{N}$ such that $\tilde{N} \geq N$ and $N + \tilde{N}$ is even.

The symbols of the scaling function ϕ and $\tilde{\phi}$ are defined by

$$m(\omega) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h_n e^{-in\omega}, \quad \tilde{m}(\omega) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \tilde{h}_n e^{-in\omega}.$$

Since $h_n = 2^{-N+1/2} \binom{N}{n}$ for $n = 0, \ldots N$, and $h_n = 0$ otherwise, we have

$$m(\omega) = \sum_{n=0}^{N} \frac{1}{2^N} \binom{N}{n} e^{-in\omega} = \frac{\left(e^{-i\omega} + 1\right)^N}{2^N}.$$

Lemma: The biorthogonality of ϕ and $\tilde{\phi}$ implies

$$\overline{m(\omega)}\tilde{m}(\omega) + \overline{m(\omega+\pi)}\tilde{m}(\omega+\pi) = 1.$$

Thus, for the given symbol $m(\omega)$ we find a trigonometric polynomial $\tilde{m}(\omega)$ such that the above identity is satisfied.

Lemma: For $M \in \mathbb{N}$ let us define a polynomial

$$p_M(x) = \sum_{n=0}^{M-1} \binom{M-1+n}{n} x^n.$$

Then $(1-x)^M p_M(x) + x^M p_M(1-x) = 1$ for all $x \in \mathbb{R}$. Replacing x by $\sin^2 \frac{\omega}{2}$, we obtain

$$\left(\cos^2 \frac{\omega}{2}\right)^M p_M\left(\sin^2 \frac{\omega}{2}\right) + \left(\cos^2 \frac{\omega+\pi}{2}\right)^M p_M\left(\sin^2 \frac{\omega+\pi}{2}\right) = 1.$$

Therefore, it is sufficient to find trigonometric polynomials satisfying

$$\overline{m(\omega)}\tilde{m}(\omega) = \left(\cos^2\frac{\omega}{2}\right)^M p_M\left(\sin^2\frac{\omega}{2}\right).$$

We set $M = \frac{N+\tilde{N}}{2}$ and we replace $e^{i\omega}$ by z. The symbol of ϕ satisfies

$$\overline{m(\omega)} = \frac{\left(e^{i\omega}+1\right)^N}{2^N} = \left(\frac{z+1}{2}\right)^N.$$

We have

$$\cos^2\frac{\omega}{2} = \left(\frac{e^{i\omega/2}+e^{-i\omega/2}}{2}\right)^2 = \left(\frac{\sqrt{z}+\frac{1}{\sqrt{z}}}{2}\right)^2 = \frac{(z+1)^2}{4z}.$$

Thus, the scaling coefficients $\tilde{h}_n$ of the dual scaling function are given by:

$$\left(\frac{z+1}{2}\right)^N \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}\tilde{h}_n z^{-n} = \frac{(z+1)^{2M}}{4^M z^M}p_M\left(1 - \frac{(z+1)^2}{4z}\right).$$

The wavelet filters are given by

$$g_n = (-1)^n \tilde{h}_{1-n}, \quad \tilde{g}_n = (-1)^n h_{1-n},$$

and wavelets are given by

$$\psi(x) = \sum_{n \in \mathbb{Z}} g_n \sqrt{2} \phi(2x - n), \quad \tilde{\psi}(x) = \sum_{n \in \mathbb{Z}} \tilde{g}_n \sqrt{2} \tilde{\phi}(2x - n).$$

Theorem: Functions ϕ and ψ generates a wavelet basis Ψ of the space $L^2(\mathbb{R})$, and wavelets have $\tilde{N}$ vanishing moments. Functions $\tilde{\phi}$ and $\tilde{\psi}$ generate a wavelet basis $\tilde{\Psi}$ of the space $L^2(\mathbb{R})$, which is biorthogonal to Ψ and wavelets have N vanishing moments.

Construction

1. Choose the order of spline N and the number of vanishing moments $\tilde{N}$ such that $\tilde{N} \geq N$ and $N + \tilde{N}$ is even. Set $\phi = B_N$ and compute $h_n = 2^{-N+1/2} \binom{N}{n}$ for $n = 0, \ldots N$.

2. Set $M = (N + \tilde{N})/2$ and compute $p_M(x) = \sum_{n=0}^{M-1} \binom{M-1+n}{n} x^n$.

3. Compute scaling coefficients $\tilde{h}_n$ using

$$\left(\frac{z+1}{\sqrt{2}} \right)^N \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{h}_n z^{-n} = \frac{(z+1)^{2M}}{4^M z^M} p_M \left(1 - \frac{(z+1)^2}{4z} \right).$$

4. Compute wavelet filters

$$g_n = (-1)^n \tilde{h}_{1-n}, \quad \tilde{g}_n = (-1)^n h_{1-n}.$$

5. Primal scaling functions and wavelets are splines and are given explicitly. Dual scaling functions and wavelets are given by scaling and wavelet equations.

Example 6. Let $N = 3$ and $\tilde{N} = 5$. Scaling filter is given by
$\mathbf{h} = \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4} \right) / \sqrt{2}$.
Then $M = \frac{N + \tilde{N}}{2} = 4$ and

$$
p_M(x) = \sum_{n=0}^{M-1} \binom{M-1+n}{n} x^n = 1 + 4x + 10x^2 + 20x^3
$$

and the scaling coefficients $\tilde{h}_n$ of the dual scaling function are given by:

$$
\left(\frac{z+1}{\sqrt{2}} \right)^N \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{h}_n z^{-n} = \frac{(z+1)^{2M}}{4^M z^M} p_M \left(1 - \frac{(z+1)^2}{4z} \right).
$$

We obtain

$$\sqrt{2}\sum_{n\in\mathbb{Z}}\tilde{h}_n z^{-n} = \frac{(z+1)^5}{2^4 z^4}\left(\frac{-5}{16z^3}+\frac{5}{2z^2}-\frac{131}{16z}+13-\frac{131z}{16}+\frac{5z^2}{2}-\frac{5z^3}{16}\right)$$

$$= -\frac{5}{256}z^{-7}+\frac{15}{256}z^{-6}+\frac{19}{256}z^{-5}-\frac{97}{256}z^{-4}-\frac{13}{128}z^{-3}+\frac{175}{128}z^{-2}$$

$$+ \frac{175}{128}z^{-1}-\frac{13}{128}-\frac{97}{256}z+\frac{19}{256}z^2+\frac{15}{256}z^3-\frac{5}{256}z^4.$$

Hence, we have

$$\tilde{h} = \frac{1}{\sqrt{2}}\left(\frac{-5}{256},\frac{15}{256},\frac{19}{256},\frac{-97}{256},\frac{-13}{128},\frac{175}{128},\frac{175}{128},\frac{-13}{128},\frac{-97}{256},\frac{19}{256},\frac{15}{256},\frac{-5}{256}\right).$$

Table: Scaling coefficients of primal and dual scaling functions for several values of parameters N and $\tilde{N}$.

N	$\sqrt{2}\{h_n\}$	$\tilde{N}$	$\sqrt{2}\left\{\tilde{h}_n\right\}$
1	$\{1,1\}$	1	$\{1,1\}$
		3	$\left\{\frac{-1}{8}, \frac{1}{8}, 1, 1, \frac{1}{8}, \frac{-1}{8}\right\}$
		5	$\left\{\frac{3}{128}, \frac{-3}{128}, \frac{-11}{64}, \frac{11}{64}, 1, 1, \frac{11}{64}, \frac{-11}{64}, \frac{-3}{128}, \frac{3}{128}\right\}$
2	$\left\{\frac{1}{2}, 1, \frac{1}{2}\right\}$	2	$\left\{\frac{-1}{4}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-1}{4}\right\}$
		4	$\left\{\frac{3}{64}, \frac{-3}{64}, \frac{-1}{4}, \frac{19}{32}, \frac{45}{32}, \frac{19}{32}, \frac{-1}{4}, \frac{-3}{64}, \frac{3}{64}\right\}$
3	$\left\{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right\}$	3	$\left\{\frac{3}{32}, \frac{-9}{32}, \frac{-7}{32}, \frac{45}{32}, \frac{45}{32}, \frac{-7}{32}, \frac{-9}{32}, \frac{3}{32}\right\}$
		5	$\left\{\frac{-5}{256}, \frac{15}{256}, \frac{19}{256}, \frac{-97}{256}, \frac{-13}{128}, \frac{175}{128}, \frac{175}{128}, \frac{-13}{128}, \frac{-97}{256}, \frac{19}{256}, \frac{15}{256}, \frac{-5}{256}\right\}$

The Sobolev regularity γ of a function f is defined by

$$\gamma := \sup \left\{ s : f \in H^s\left(\mathbb{R}\right) \right\},$$

where $H^s\left(\mathbb{R}\right)$ denotes the standard Sobolev space.

The Sobolev regularity of the primal scaling function $\phi = B_N$ is $\gamma = N - \frac{1}{2}$. The Sobolev regularity of the dual scaling functions can be computed by the algorithm from [Eirola, 1992].
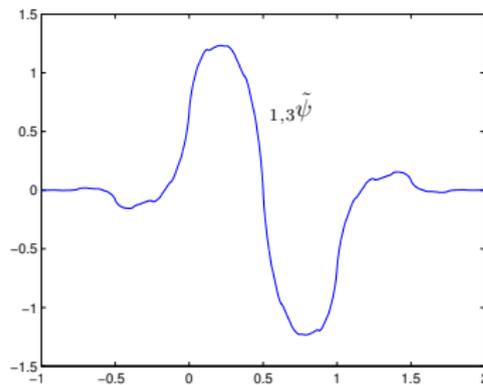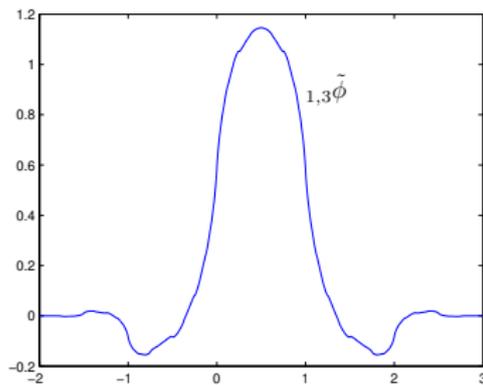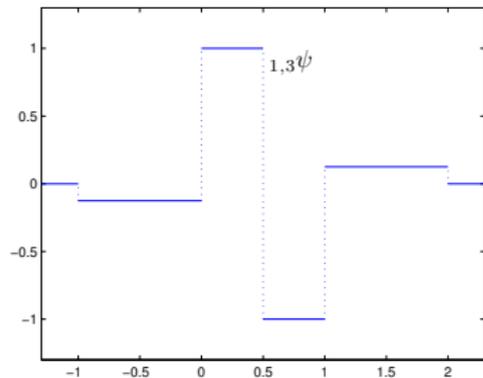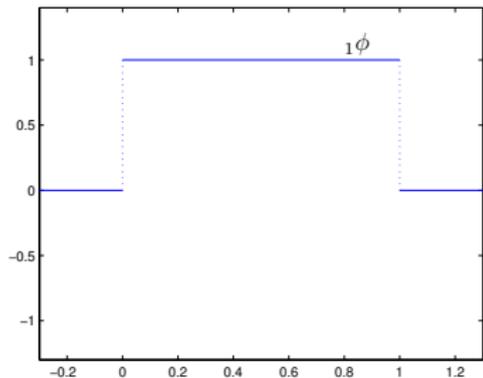
Table: Sobolev exponent of smoothness $\tilde{\gamma}$ of the dual scaling function $\tilde{\phi}$.

N	$\tilde{N}$	$\tilde{\gamma}$	N	$\tilde{N}$	$\tilde{\gamma}$	N	$\tilde{N}$	$\tilde{\gamma}$
2	2	0.441	3	3	0.175	4	6	0.344
2	4	1.175	3	5	0.793	4	8	0.862
2	6	1.793	3	7	1.344	4	10	1.363

Biorhogonal scaling functions and wavelets for $N = 1$ and $\tilde{N} = 1$
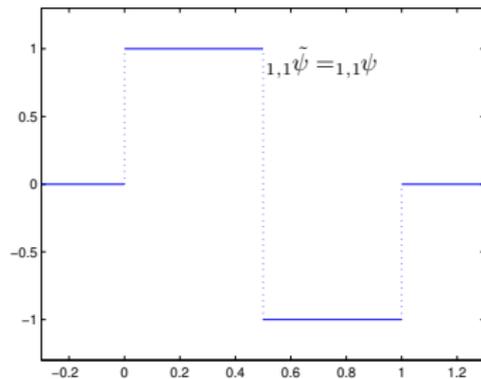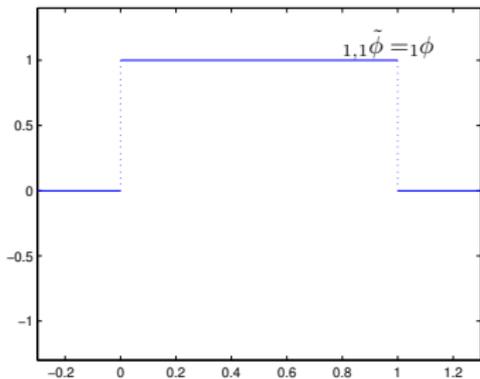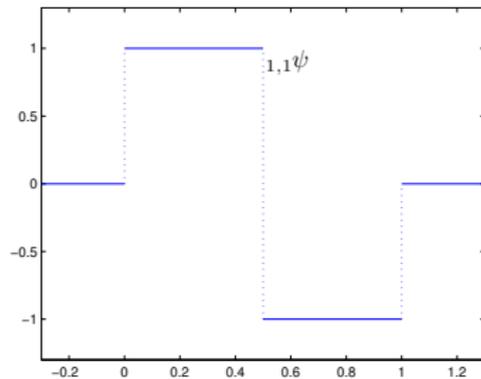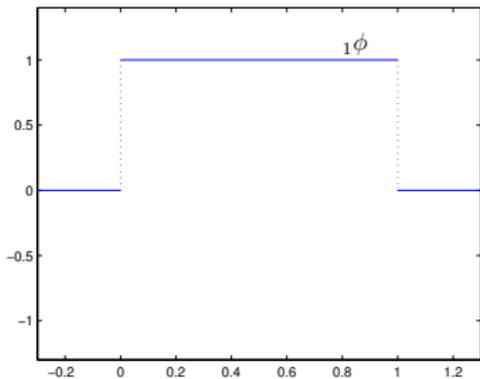
Biorhogonal scaling functions and wavelets for $N = 1$ and $\tilde{N} = 3$

Biorhogonal scaling functions and wavelets for $N = 2$ and $\tilde{N} = 4$

Biorhogonal scaling functions and wavelets for $N = 2$ and $\tilde{N} = 6$
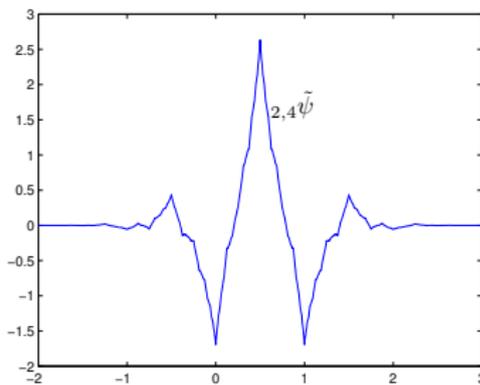
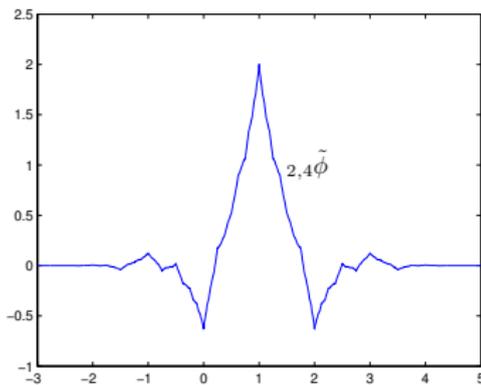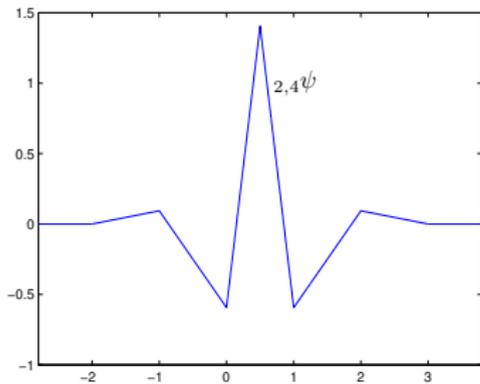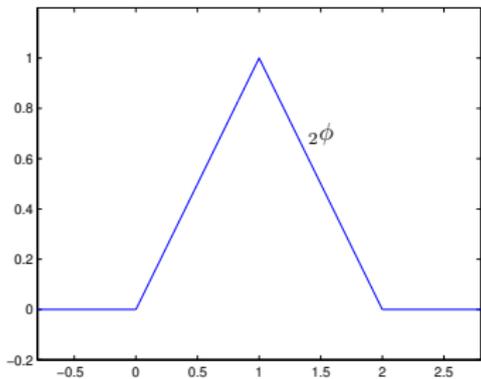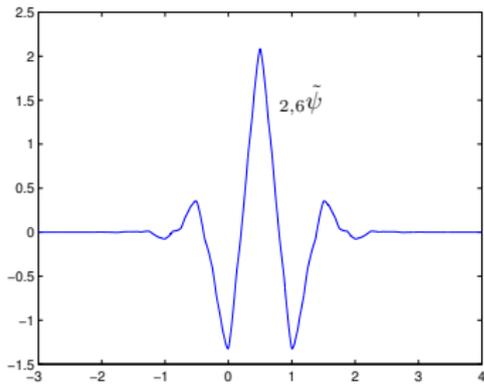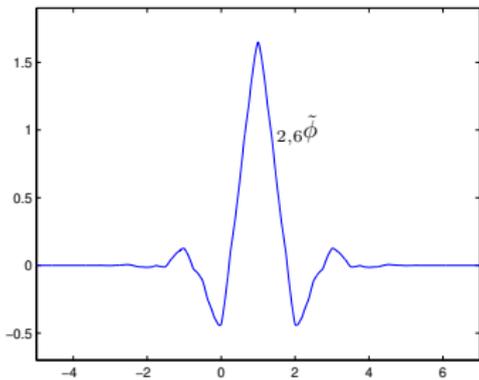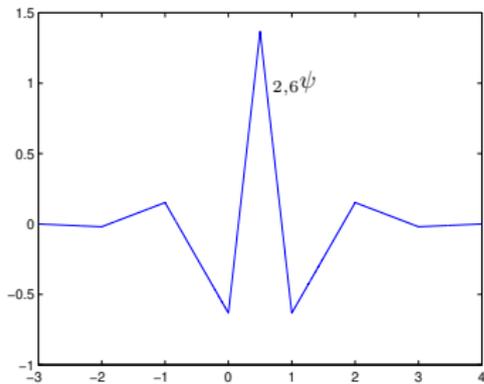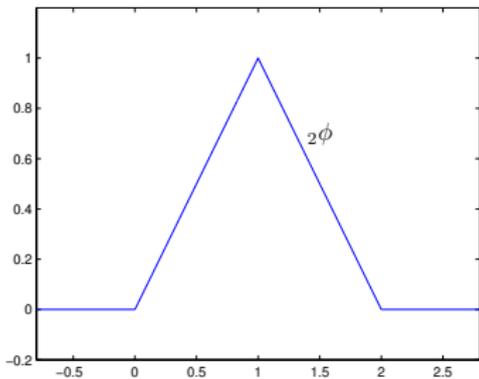Biorhogonal scaling functions and wavelets for $N = 3$ and $\tilde{N} = 5$
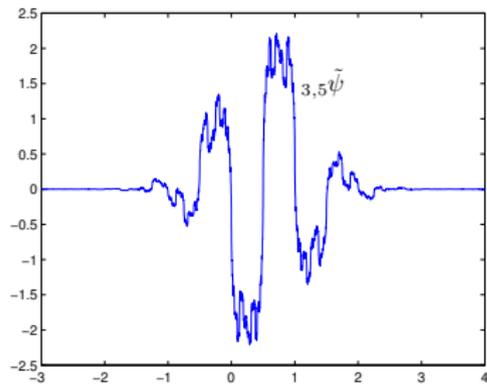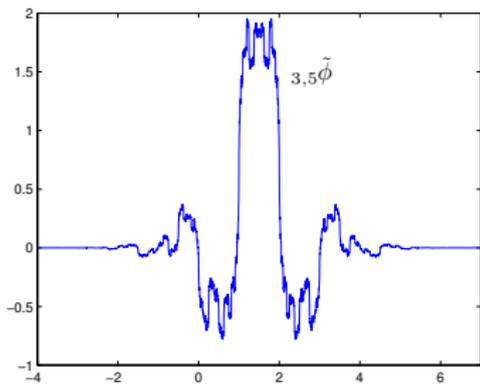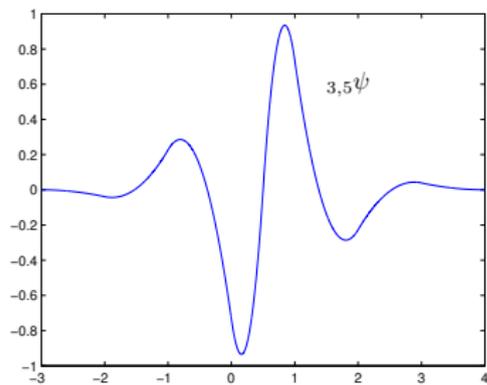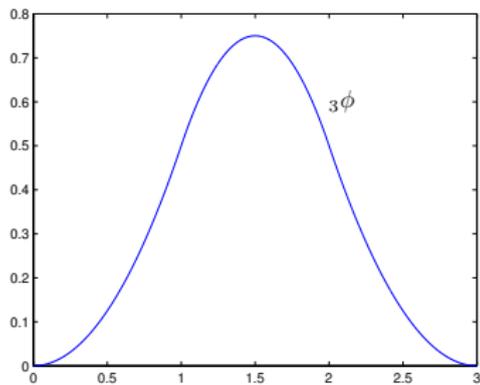
Biorhogonal scaling functions and wavelets for $N = 4$ and $\tilde{N} = 6$
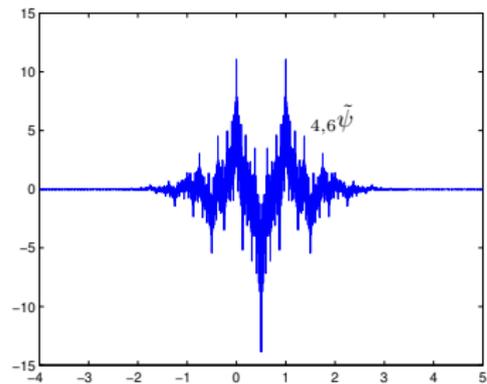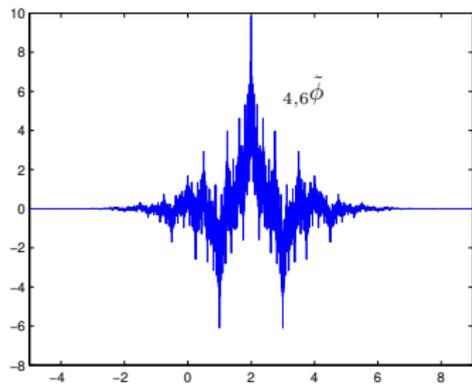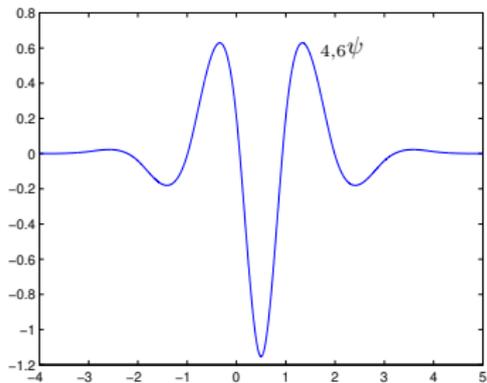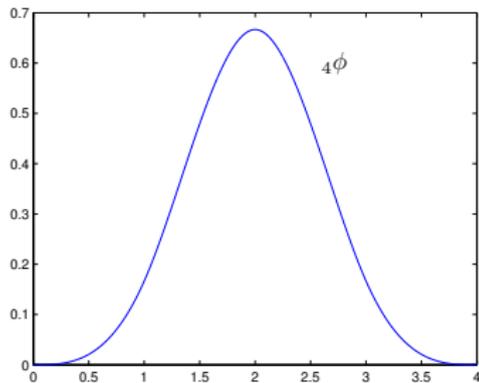
Wavelet bases on the interval

Let H be a Sobolev space or the L^2–space, $\mathcal{J}$ be an index set and let $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$. A wavelet basis of H is defined as a family $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\}$ such that

i) Ψ is a Riesz basis for H, i.e. the closure of the span of Ψ is H and there exist constants $c, C \in (0, \infty)$ such that

$$c \, \|\mathbf{b}\|_2 \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \leq C \, \|\mathbf{b}\|_2 \,,$$

for all $\mathbf{b} = \{b_\lambda\}_{\lambda \in \mathcal{J}}$ such that $\sum_{\lambda \in \mathcal{J}} b_\lambda^2 < \infty$, $\inf C / \sup c$ is called the condition number of Ψ.

ii) The functions are local in the sense that $\operatorname{diam} \operatorname{supp} \psi_\lambda \leq C 2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, and at a given level j the supports of only finitely many wavelets overlap at any point x.

A wavelet basis on the interval I has typically the hierarchical structure:

$$\Psi^I = \Phi_{j_0}^I \cup \bigcup_{j=j_0}^{\infty} \Psi_j^I.$$

$\Phi_{j_0}^I = \left\{ \phi_{j_0,k}^I, k \in \mathcal{I}_{j_0} \right\}$ - the set of scaling functions

$\Psi_j^I = \left\{ \psi_{j,k}^I, k \in \mathcal{J}_j \right\}$ - the set of wavelets

Wavelets and scaling functions in the inner part of the interval are typically translations and dilations of one or several functions. Wavelets and scaling functions near the boundary are dilations of some special functions called boundary scaling functions and boundary wavelets.

We assume that wavelets have vanishing moments, i.e.

$$\int_I x^m \psi_{j,k}(x)\, \mathrm{d}x = 0, \quad m = 0, \dots L-1, \quad k \in \mathcal{J}_j,$$

where $L \geq 1$ is dependent on the type of a wavelet.

Wavelet bases on product domains

Let Ψ^I and Ψ^J be wavelet bases on intervals I and J, respectively. A wavelet basis Ψ on the rectangle $I \times J$ can be constructed by an anisotropic approach, i.e. $\Psi = \Psi^I \otimes \Psi^J$, where $\otimes$ denotes a tensor product, or by an isotropic approach:

$$\Psi = F_{j_0} \cup \bigcup_{j=j_0}^{\infty} \left(G_j^1 \cup G_j^2 \cup G_j^3 \right),$$

where

$$
\begin{aligned}
F_j &= \left\{ \phi_{j,k}^I \otimes \phi_{j,l}^J \,/\, \left\| \phi_{j,k}^I \otimes \phi_{j,l}^J \right\|_H , k \in \mathcal{I}_I, l \in \mathcal{I}_J \right\}, \\
G_j^1 &= \left\{ \phi_{j,k}^I \otimes \psi_{j,l}^J \,/\, \left\| \phi_{j,k}^I \otimes \psi_{j,l}^J \right\|_H , k \in \mathcal{I}_I, l \in \mathcal{J}_J \right\}, \\
G_j^2 &= \left\{ \psi_{j,k}^I \otimes \phi_{j,l}^J \,/\, \left\| \psi_{j,k}^I \otimes \phi_{j,l}^J \right\|_H , k \in \mathcal{J}_I, l \in \mathcal{I}_J \right\}, \\
G_j^3 &= \left\{ \psi_{j,k}^I \otimes \psi_{j,l}^J \,/\, \left\| \psi_{j,k}^I \otimes \psi_{j,l}^J \right\|_H , k \in \mathcal{J}_I, l \in \mathcal{J}_J \right\}.
\end{aligned}
$$

Quadratic spline wavelets satisfying homogeneous boundary conditions [Černá, Finěk, 2018]

The objective is to construct a wavelet basis on $\Omega_d = (0,1)^d$, $d = 1, 2$, that satisfies the following properties:

Riesz basis property. We construct Riesz bases of $H_0^1(\Omega_d)$.

Locality. The primal basis functions are local.

Vanishing moments. The wavelets have one vanishing moment.

Polynomial exactness. Since the basis functions are quadratic splines, the primal multiresolution analysis has polynomial exactness of order three.

Short support. The wavelets have the shortest possible support among quadratic spline wavelets with one vanishing moment.

Closed form. The basis functions and wavelets have an explicit expression.

Homogeneous Dirichlet boundary conditions. The wavelet basis satisfies homogeneous Dirichlet boundary conditions of the first order.

Well-conditioned bases. The wavelet basis is well-conditioned.

Let ϕ be a quadratic B-spline defined on knots $[0, 1, 2, 3]$. It can be written explicitly as:

$$\phi(x) = \begin{cases} \frac{x^2}{2}, & x \in [0, 1], \\ -x^2 + 3x - \frac{3}{2}, & x \in [1, 2], \\ \frac{x^2}{2} - 3x + \frac{9}{2}, & x \in [2, 3], \\ 0, & \text{otherwise.} \end{cases}$$

Let ϕ_b be a multiple of the quadratic B-spline defined on knots $[0, 0, 1, 2]$ such that $\|\phi_b\|_{L^1} = \|\phi\|_{L^1}$, i.e.

$$\phi_b(x) = \begin{cases} -\frac{9x^2}{4} + 3x, & x \in [0, 1], \\ \frac{3x^2}{4} - 3x + 3, & x \in [1, 2], \\ 0, & \text{otherwise.} \end{cases}$$
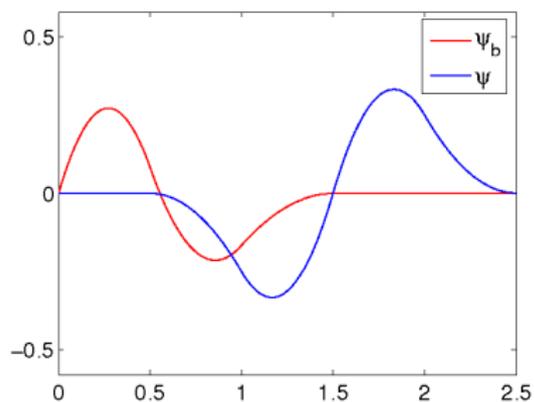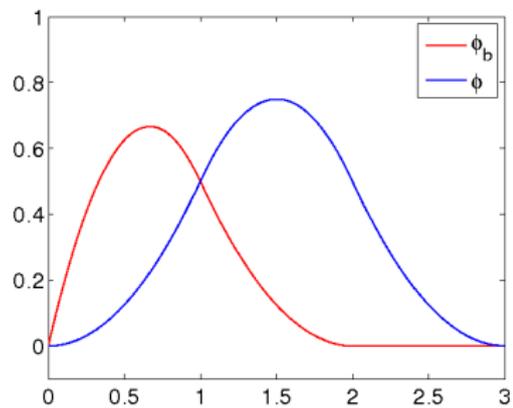
Figure: Scaling functions (left) and wavelets (right).

For $j \geq 2$ and $x \in [0, 1]$ we set

$$
\begin{aligned}
\phi_{j,k}(x) &= 2^{j/2}\phi(2^j x - k + 2), \quad k = 2, ..., 2^j - 1, \\
\phi_{j,1}(x) &= 2^{j/2}\phi_b(2^j x), \quad \phi_{j,2^j}(x) = 2^{j/2}\phi_b(2^j(1 - x)).
\end{aligned}
$$

Scaling functions on the level $j = 2$.
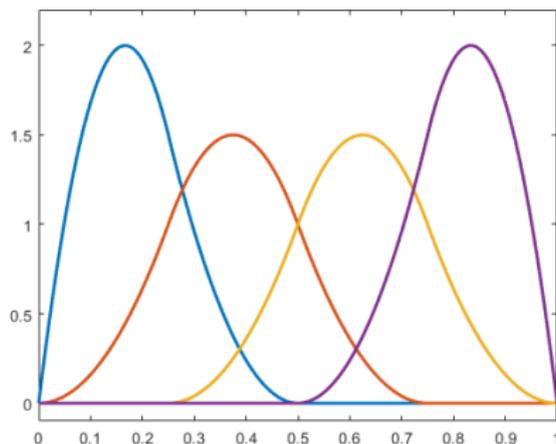
We define a wavelet ψ and a boundary wavelet ψ_b as

$$\psi(x) = -\frac{1}{2}\phi(2x-1) + \frac{1}{2}\phi(2x-2) \quad \text{and} \quad \psi_b(x) = \frac{-\phi_b(2x)}{2} + \frac{\phi(2x)}{2}.$$

Then $\operatorname{supp}\psi = [0.5, 2.5]$, $\operatorname{supp}\psi_b = [0, 1.5]$, and both wavelets have one vanishing moment, i.e.

$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \psi_b(x)dx = 0.$$

The wavelets ψ and ψ_b have the shortest possible support.

Lemma. Let ϕ be defined as above. If
$\psi \in \overline{\text{span}} \{\phi (2 \cdot -k), k \in \mathbb{Z}\}$ and ψ has one vanishing moment,
then the length of the support of ψ is at least two. Similarly, the
boundary wavelet ψ_b has the shortest possible support among all
boundary wavelets with one vanishing moment generated from
scaling functions ψ and ψ_b.

For $j \geq 2$ and $x \in [0, 1]$ we define

$$
\begin{aligned}
\psi_{j,k}(x) &= 2^{j/2}\psi(2^j x - k + 2), k = 2, ..., 2^j - 1, \\
\psi_{j,1}(x) &= 2^{j/2}\psi_b(2^j x), \quad \psi_{j,2^j}(x) = -2^{j/2}\psi_b(2^j(1-x)).
\end{aligned}
$$

We denote the index sets by

$$
\mathcal{I}_j = \left\{ k \in \mathbb{Z} : 1 \leq k \leq 2^j \right\}.
$$

We define

$$
\Phi_j = \{\phi_{j,k}, k \in \mathcal{I}_j\}, \quad \Psi_j = \{\psi_{j,k}, k \in \mathcal{I}_j\},
$$

and

$$
\Psi = \Phi_2 \cup \bigcup_{j=2}^{\infty} \Psi_j, \quad \Psi^s = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j, \quad j_0 = 2.
$$

We use the isotropic approach for the construction of a wavelet basis Ψ^{2D} on $\Omega_2 = (0,1)^2$.
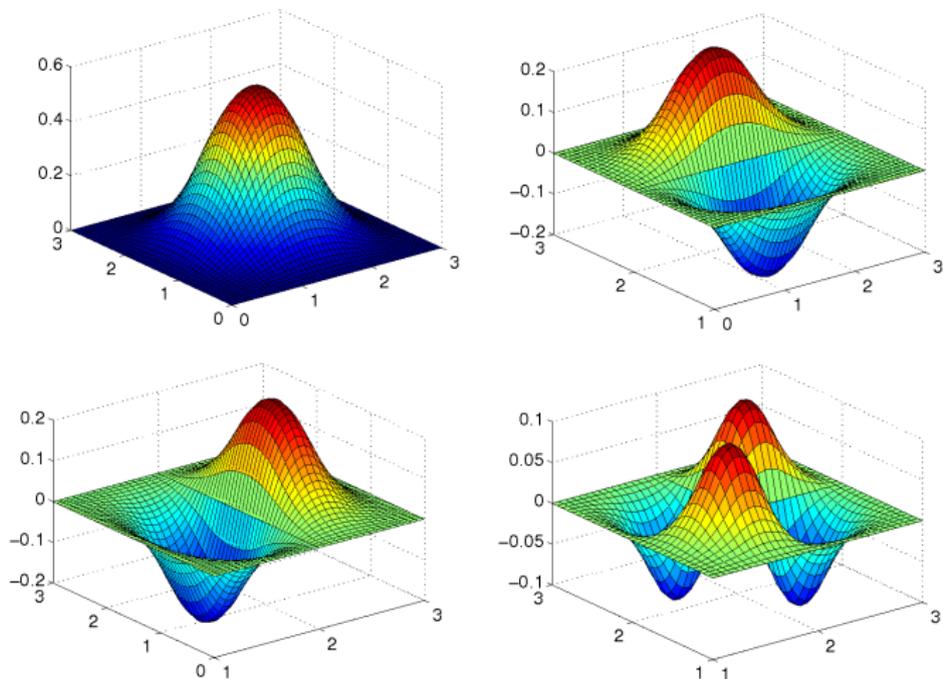
Figure: Scaling function $\phi \otimes \phi$ (upper left) and wavelets $\phi \otimes \psi$ (upper right), $\psi \otimes \phi$ (lower left), and $\psi \otimes \psi$ (lower right).

Let $|\cdot|_{H_0^1(0,1)}$ denotes the $H_0^1(0,1)$–seminorm, i.e.

$$|f|_{H_0^1(0,1)} = \sqrt{\int\limits_0^1 (f'(x))^2\, dx}.$$

Theorem. The set Ψ when normalized with respect to the H^1-seminorm, i.e. the set

$$\left\{ \phi_{2,k}/\left|\phi_{2,k}\right|_{H_0^1(0,1)}, k \in \mathcal{I}_2 \right\} \cup \left\{ \psi_{j,k}/\left|\psi_{j,k}\right|_{H_0^1(0,1)}, j \geq 2, k \in \mathcal{I}_j \right\},$$

is a Riesz basis of $H_0^1(0,1)$.

Theorem. The set Ψ^{2D} normalized with respect to the H^1–seminorm is a Riesz basis of $H_0^1\left((0,1)^2\right)$.

Quantitative properties

We present the condition numbers of the stiffness matrices for the Helmholtz equation

$$- \epsilon \Delta u + a u = f \ \text{ on } \ \Omega_d, \quad u = 0 \text{ on } \partial \Omega_d,$$

where Δ is the Laplace operator, ϵ and a are positive constants. We also study the case $\epsilon = 1$ and $a = 0$, i.e. the Poisson equation, and the case $\epsilon = 0$ and $a = 1$.

An advantage of discretizing elliptic equations using a wavelet basis is that the discrete system can be simply preconditioned by a diagonal preconditioner [Dahmen, Kunoth, 1992].

Let Ψ^s be a wavelet basis with s levels of wavelets. We define

$$\mathbf{A}_s = \epsilon \left\langle \nabla \Psi^s, \nabla \Psi^s \right\rangle + a \left\langle \Psi^s, \Psi^s \right\rangle, \quad u_s = \left(\mathbf{u}_s \right)^T \Psi^s, \quad \mathbf{f}_s = \left\langle f, \Psi^s \right\rangle.$$

Let $\mathbf{D}_s$ be a matrix of diagonal elements of the matrix $\mathbf{A}_s$, i.e. $\left(\mathbf{D}_s \right)_{\lambda,\mu} = \left(\mathbf{A}_s \right)_{\lambda,\mu} \delta_{\lambda,\mu}$. We set

$$\tilde{\mathbf{A}}_s = \left(\mathbf{D}_s \right)^{-1/2} \mathbf{A}_s \left(\mathbf{D}_s \right)^{-1/2}, \quad \tilde{\mathbf{u}}_s = \left(\mathbf{D}_s \right)^{1/2} \mathbf{u}_s, \quad \tilde{\mathbf{f}}_s = \left(\mathbf{D}_s \right)^{-1/2} \mathbf{f}_s$$

and we obtain the preconditioned finite-dimensional system

$$\tilde{\mathbf{A}}_s \tilde{\mathbf{u}}_s = \tilde{\mathbf{f}}_s.$$

The matrix $\tilde{\mathbf{A}}_s$ is symmetric and positive definite and the condition numbers are uniformly bounded, i.e.

$$\operatorname{cond} \tilde{\mathbf{A}}_s \leq C,$$

where C is a constant independent on s.

The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ correspond to the squares of the condition numbers of Ψ^s with respect to the H^1-seminorm.

Table: The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ of the size $N \times N$ corresponding to multiscale wavelet bases with s levels of wavelets for the one-dimensional and the two-dimensional Poisson equation.

		1D				2D		
s	N	λ_{min}	λ_{max}	$\mathrm{cond}\tilde{\mathbf{A}}_s$	N	λ_{min}	λ_{max}	$\mathrm{cond}\tilde{\mathbf{A}}_s$
1	8	0.50	1.38	2.77	64	0.25	1.88	7.5
2	16	0.50	1.41	2.83	256	0.19	2.08	11.1
3	32	0.50	1.42	2.83	1 024	0.16	2.17	13.7
4	64	0.50	1.42	2.84	4 096	0.14	2.20	15.4
5	128	0.50	1.42	2.84	16 384	0.13	2.22	16.6
6	256	0.50	1.42	2.84	65 536	0.13	2.23	17.4
7	512	0.50	1.42	2.84	262 144	0.12	2.23	17.9
8	1024	0.50	1.42	2.84	1 048 576	0.12	2.23	18.3

Table: The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ of the size $N \times N$ corresponding to multiscale wavelet bases with s levels of wavelets for the three-dimensional Poisson equation.

s	N	λ_{min}	λ_{max}	$\mathrm{cond}\tilde{\mathbf{A}}_s$
1	512	0.15	3.23	47.4
2	4096	0.04	3.69	85.0
3	32768	0.03	3.83	113.8
4	262144	0.03	3.87	132.9
5	2097152	0.03	3.89	145.3

Table: The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ of the size 65536×65536 for several choices of ϵ and a.

ϵ	a	CF_2	CF_3	CF_2^{ort}	CF_3^{ort}	CQ	D_2	D_3
1000	1	17.4	16.3	17.1	16.4	62.0	116.3	98.4
1	0	17.4	16.7	17.1	16.4	62.0	116.3	98.4
1	1	17.4	16.7	17.1	16.4	62.0	116.6	98.5
10^{-3}	1	72.1	35.9	35.6	22.5	61.1	328.1	139.2
10^{-6}	1	746.0	577.0	425.7	287.6	46.3	1878.0	1115.4
0	1	872.6	687.4	511.0	351.5	46.4	2034.6	1251.4

CF_{j_0} - the constructed basis with the coarsest level j_0, $CF_{j_0}^{ort}$ - the constructed basis with orthogonalization of scaling functions on the coarsest level, CQ - Chui-Quak semiorthogonal spline wavelets, D_{j_0} - Dijkema spline wavelets with the coarsest level j_0
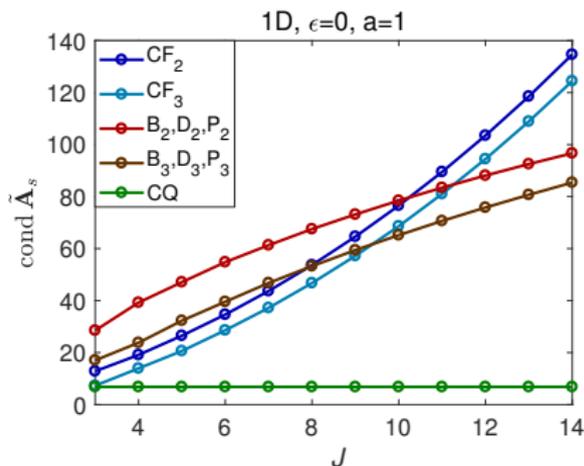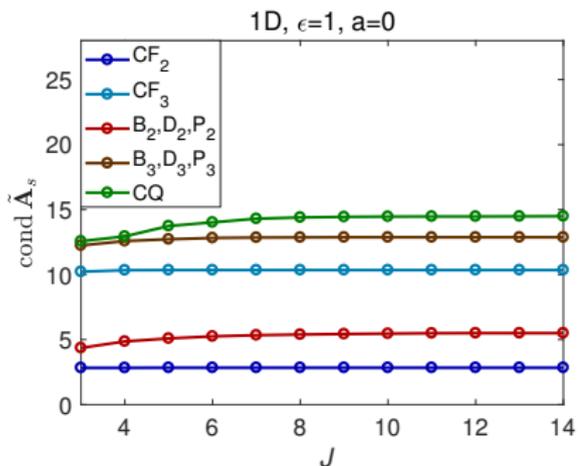
Figure: The condition numbers of the matrices $\tilde{\mathbf{A}}_s$, $s = J - j_0 + 1$, for the Helmholtz equation with parameters $\epsilon = 1$, $a = 0$, and $\epsilon = 0$, $a = 1$. The parameter J denotes the finest level and j_0 denotes the coarsest level.
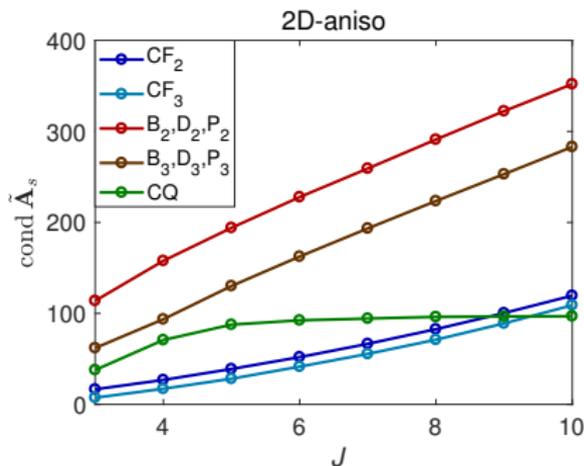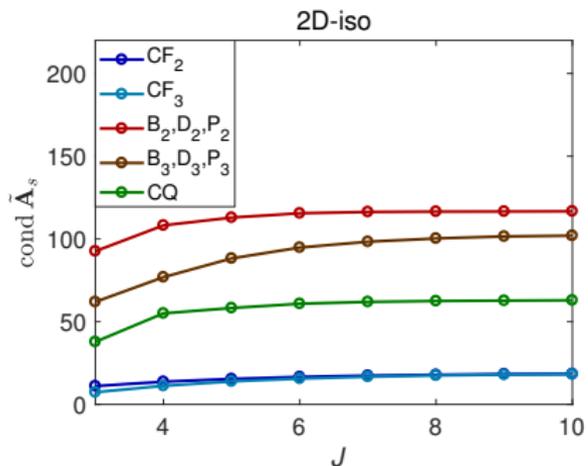
Figure: The condition numbers of the matrices $\tilde{\mathbf{A}}_s$, $s = J - j_0 + 1$, for $\epsilon = 1$, $a = 0$ and two-dimensional wavelet bases constructed using an isotropic approach and an anisotropic approach. The parameter J denotes the finest level and j_0 denotes the coarsest level.

Multilevel Galerkin method

We consider the Helmholz equation for Ω_2, $\epsilon = 1$ and $a = 0$. The right-hand side f is such that the solution u is given by

$$u(x, y) = v(x) v(y), \quad v(x) = x \left(1 - e^{50x - 50}\right).$$

We discretize the equation using the Galerkin method with the constructed wavelet basis and we obtain discrete problem $\tilde{\mathbf{A}}_s \tilde{\mathbf{u}}_s = \tilde{\mathbf{f}}_s$. We solve it by conjugate gradient method using a simple multilevel approach.

1. Compute $\tilde{\mathbf{A}}_s$ and $\tilde{\mathbf{f}}_s$, choose $\mathbf{v}_0$ of the length 4^2.

2. For $j = 0, \ldots, s$ find the solution $\tilde{\mathbf{u}}_j$ of the system $\tilde{\mathbf{A}}_j \tilde{\mathbf{u}}_j = \tilde{\mathbf{f}}_j$ by conjugate gradient method with initial vector $\mathbf{v}_j$ defined for $j \geq 1$ by

$$(\mathbf{v}_j) = \begin{cases} \tilde{\mathbf{u}}_{j-1}, & i = 1, \ldots, k_j, \\ 0, & i = k_j, \ldots, k_{j+1}, \end{cases}$$

where $k_j = 2^{2(j+1)}$.

Let u_s be an approximate solution obtained by multilevel Galerkin method with s levels of wavelets.

Theorem: If we use the criterion for terminating iterations $\|\mathbf{r}_s\|_2 \leq C2^{-2s}$, where $\mathbf{r}_s := \tilde{\mathbf{A}}_s \tilde{\mathbf{u}}_s - \tilde{\mathbf{f}}_s$, then

$$\|u - u_s\| \leq C2^{-3s}, \quad \|u - u_s\|_{H^1(\Omega_d)} \leq C2^{-2s}.$$

For the given number of levels s we used the criterion $\|\mathbf{r}_j\|_2 \leq 10^{-4}2^{-2s}$, $j = 0, \ldots, s$, for terminating iterations in each level.

We denote the number of iterations on the level j as M_j. One CG iteration can be performed with complexity of the order $\mathcal{O}(N)$, where $N \times N$ is the size of the matrix. Therefore the number of operations needed to compute one CG iteration on the level j requires about one quarter of operations needed to compute one CG iteration on the level $j + 1$, we compute the total number of equivalent iterations by

$$M = \sum_{j=0}^{s} \frac{M_j}{4^{s-j}}.$$

We compute experimental rates of convergence r_2 and r_∞ as

$$r_2 = \frac{\log\left(\|u_{s-1} - u\| \, / \, \|u_s - u\|\right)}{\log 2},$$

$$r_\infty = \frac{\log\left(\|u_{s-1} - u\|_\infty \, / \, \|u_s - u\|_\infty\right)}{\log 2}.$$

We present also the wall clock time. It includes the computation of the right-hand side, the system matrix, iterations and evaluation of the solution on the grid with the step size 2^{-j_0-s}, where j_0 is the coarsest level.

Table: Number of iterations, error estimates, and computational time for multilevel conjugate gradient method.

s	N	M	$\|u_s - u\|_\infty$	r_∞	$\|u_s - u\|$	r_2	time [s]
1	64	18.50	3.19e-1		4.54e-2		0.04
2	256	21.63	1.32e-1	1.27	1.26e-3	5.17	0.05
3	1 024	23.66	2.60e-2	2.34	2.02e-3	2.64	0.06
4	4 096	23.00	2.91e-3	3.16	2.45e-4	3.04	0.09
5	16 384	20.89	4.06e-4	2.84	2.89e-5	3.08	0.16
6	65 536	18.37	5.35e-5	2.92	3.41e-6	3.08	0.30
7	262 144	15.68	6.82e-6	2.97	4.23e-7	3.01	0.99
8	1 048 576	13.02	8.63e-7	2.98	5.28e-8	3.00	3.89
9	4 194 304	10.35	1.08e-7	3.00	6.59e-9	3.00	14.87
10	16 777 216	8.85	1.41e-8	2.94	8.25e-10	3.00	58.12

Table: Error estimates for multilevel conjugate gradient method.

s	CF $\|u_s - u\|_\infty$	Primbs, Dijkema $\|u_s - u\|_\infty$	Chui and Quak $\|u_s - u\|_\infty$
1	3.19e-1	3.19e-1	3.19e-1
2	1.32e-1	1.32e-1	1.32e-1
3	2.60e-2	2.60e-2	2.60e-2
4	2.91e-3	2.91e-3	2.91e-3
5	4.06e-4	4.06e-4	4.06e-4
6	5.35e-5	5.35e-5	5.35e-5
7	6.82e-6	6.84e-6	6.84e-6
8	8.63e-7	8.64e-7	8.64e-7
9	1.08e-7	1.09e-7	1.08e-7
10	1.41e-8	1.38e-8	1.36e-8

Table: Number of iterations and computational time for multilevel
conjugate gradient method.

s	CF M	time [s]	Primbs, Dijkema M	time [s]	Chui and Quak M	time [s]
1	18.50	0.04	27.50	0.04	13.00	0.03
2	21.63	0.05	48.88	0.07	30.25	0.05
3	23.66	0.06	59.22	0.11	35.06	0.07
4	23.00	0.09	59.38	0.19	33.82	0.14
5	20.89	0.16	50.76	0.33	30.30	0.21
6	18.37	0.30	39.44	0.68	25.32	0.41
7	15.68	0.99	29.92	2.20	20.74	1.39
8	13.02	3.89	21.50	9.53	17.87	5.55
9	10.35	14.87	17.66	47.39	14.82	21.62
10	8.85	58.12	15.79	248.41	12.36	83.54

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