Wavelet Methods for Option Pricing

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Talks

1. Discrete wavelet transform, wavelets, and wavelet basis
2. Construction of spline wavelet basis
3. Wavelet methods for integro-differential equations
4. Wavelet methods for option pricing

Pdf presentation can be found at
https://kmd.fp.tul.cz/en/cb-profile/cerna

Outline

The details can be found in

D. Černá: Cubic spline wavelets with four vanishing moments on the interval and their applications to option pricing under Kou model. *International Journal of Wavelets, Multiresolution, and Information Processing* **17**, 2019, Article No. 1850061.

D. Černá: Quadratic Spline Wavelets for Sparse Discretization of Jump-Diffusion Models. *Symmetry* **11**, 2019, Article No. 999.

Introduction

- Put (call) option gives its holder the right, but not the obligation, to sell (buy) an underlying asset at a specific price (strike) on a certain date (maturity).
- Models for option pricing: the Black-Scholes model, stochastic volatility models (the Heston model, the Stein and Stein model, etc.).
- We focus on models with jumps in the price: the Merton model, the Kou model.
- These models are represented by PIDE. Most classical methods lead to full discretization matrices.
- [Beylkin, et al., 1991] Some integral operators have sparse representation in wavelet bases.
- We study the wavelet-Galerkin method combined with the Crank-Nicolson scheme for the jump-diffusion models.

Jump-diffusion models

These models assume that the price S of the asset at time $\tilde{t}$ follows the jump-diffusion process

$$\frac{dS(\tilde{t})}{S(\tilde{t}-)} = \mu\, d\tilde{t} + \sigma\, dW(\tilde{t}) + d\left(\sum_{i=1}^{N(\tilde{t})} (V_i - 1)\right),$$

where $W(\tilde{t})$ is the Wiener process (Brownian motion), σ is a volatility, μ is the drift rate, $\{V_i\}$ are i.i.d., and $N(\tilde{t})$ is the Poisson process with intensity λ, i.e.

$$P(N(\tilde{t}) = n) = \frac{(\lambda\tilde{t})^n}{n!} e^{-\lambda\tilde{t}}.$$

Merton model

$Y_i = \log V_i$ is normally distributed with the mean μ_J and the standard deviation σ_J and the density

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma_J} e^{-\frac{(x-\mu_J)^2}{2\sigma_J^2}}.$$

Kou model

$Y_i = \log V_i$ has a double exponential distribution with the density

$$g(x) = p\,\eta_1 e^{-\eta_1 x} H(x) + q\,\eta_2 e^{\eta_2 x} H(-x),$$

H - the Heaviside function, $\eta_1 > 1$, $\eta_2 > 0$, $p \in (0,1)$ - the probability of the upward jump, $q = 1 - p$ - the probability of the downward jump.

Let $t = T - \tilde{t}$ represent time to maturity, r be a risk-free rate. The value $U(S,t)$ of the option is given by [Merton, 1976; Kou, 2002]:

$$\frac{\partial U}{\partial t} - \mathcal{D}(U) - \mathcal{I}(U) = 0,\ S > 0,\ t \in (0,T),$$

where the operators $\mathcal{D}$ and $\mathcal{I}$ are given by

$$\begin{aligned} \mathcal{D}(U) &= \frac{\sigma^2 S^2}{2}\frac{\partial^2 U}{\partial S^2} + (r - \lambda\kappa) S \frac{\partial U}{\partial S} - (r+\lambda) U \\ \mathcal{I}(U) &= \lambda \int_{-\infty}^{\infty} U(Se^x) g(x)\, dx. \end{aligned}$$

($\lambda = 0$ corresponds to Black-Scholes model.) The parameter κ represents $E(V_i - 1)$, i.e.

$$\kappa = \frac{p\,\eta_1}{\eta_1 - 1} + \frac{q\,\eta_2}{\eta_2 + 1} - 1\,(\text{Kou}), \quad \kappa = \mu_J - 1\,(\text{Merton}).$$

Initial and boundary conditions

We pay our attention to a European put option. The value of a European call option can be computed using the put-call parity ($U_C - U_P = S - e^{-rt}$). The initial condition is given by $U(S,0) = \max(K - S, 0)$, where K is the strike price, and the boundary conditions are given by

$$U(0,t) = Ke^{-rt},\ U(S,t) \approx 0 \text{ for } S \to \infty.$$

We choose the minimal value S^{min} and the maximal value S^{max} and we approximate the unbounded domain $(0,\infty)$ by a domain $\Omega = \left(S^{min}, S^{max}\right)$. Since $U(S,t) \approx Ke^{-rt} - S$ for small S we replace the boundary conditions with

$$U\left(S^{min},t\right) = Ke^{-rt} - S^{min}, \quad U\left(S^{max},t\right) = 0.$$

Transformation to logarithmic prices

We transform the equation using logarithmic prices $x = \log S$. We denote $\hat{U}(x,t) = U(e^x, t)$ and we obtain

$$\frac{\partial \hat{U}}{\partial t} - \hat{\mathcal{D}}\left(\hat{U}\right) - \hat{\mathcal{I}}\left(\hat{U}\right) = 0,\ x \in X,\ t \in (0,T), \tag{1}$$

where $X = \left(x^{min}, x^{max}\right)$, $x^{min} = \log S^{min}$, $x^{max} = \log S^{max}$, and the operators $\hat{\mathcal{D}}$ and $\hat{\mathcal{I}}$ are given by

$$\hat{\mathcal{D}}\left(\hat{U}\right) = \frac{\sigma^2}{2}\frac{\partial^2 \hat{U}}{\partial x^2} + \left(r - \lambda\kappa - \frac{\sigma^2}{2}\right)\frac{\partial \hat{U}}{\partial x} - (r+\lambda)\,\hat{U}$$

and

$$\hat{\mathcal{I}}\left(\hat{U}\right) = \lambda \int_{-\infty}^{\infty} \hat{U}(x+y,t)\, g(y)\, dy = \lambda \int_{-\infty}^{\infty} \hat{U}(y,t)\, g(y-x)\, dy.$$

Approximation of integral term

Due to the decay of a value of a put option and the decay of a probability density function at infinity we have

$$\int_{x^{max}}^{\infty} \hat{U}(y,t)\, g(y-x)\, dy \approx 0 \text{ for } x^{max} \to \infty.$$

Since $U(S,t) \approx Ke^{-rt} - S$ for S close to zero, we can approximate the integral term $\hat{\mathcal{I}}$ by

$$\hat{\mathcal{I}}\left(\hat{U}\right) \approx \mathcal{I}_1 + \mathcal{I}_2\left(\hat{U}\right),$$

where

$$\mathcal{I}_1 = \int_{-\infty}^{x^{min}} \left(Ke^{-rt} - e^{x}\right) g(y-x)\, dy,\ \mathcal{I}_2\left(\hat{U}\right) = \int_{x^{min}}^{x^{max}} \hat{U}(y,t)\, g(y-x)\, dy.$$

Transformation to homogeneous boundary conditions

We transform the equation (1) to the equation with homogeneous Dirichlet boundary conditions. Let $\tilde{U} = \hat{U} - W$, where W is a function satisfying boundary conditions that is smooth enough. We can define

$$W(x,t) = \left(Ke^{-rt} - e^{x^{min}}\right)\left(1 + x^{min} - x\right)$$

for $x \in \bar{X}$ and $t \in [0,T]$. Then $\tilde{U}$ is the solution of the equation

$$\frac{\partial \tilde{U}}{\partial t} - \hat{\mathcal{D}}\left(\tilde{U}\right) - \mathcal{I}_2\left(\tilde{U}\right) = f(W),$$

with

$$f(W) = -\frac{\partial W}{\partial t} + \hat{\mathcal{D}}(W) + \mathcal{I}_2(W) + \mathcal{I}_1$$

satisfying the initial condition

$$\tilde{U}(x,0) = U(e^x,0) - W(x,0), \quad x \in X,$$

and homogeneous boundary conditions.

Variational formulation

Let a be a bilinear form defined by

$$a(u,v) = \left\langle \hat{\mathcal{D}}(u), v \right\rangle + \langle \mathcal{I}_2(u), v \rangle,$$

for all $u, v \in H_0^1(X)$.
Then the variational formulation reads as:
Find $\tilde{U} \in H_0^1\left(0, T; H_0^1(X)\right)$ such that $\frac{\partial \tilde{U}}{\partial t} \in H_0^1\left(0, T; H^{-1}(X)\right)$,
$\tilde{U}$ satisfies the initial condition and

$$\left\langle \frac{\partial \tilde{U}}{\partial t}, v \right\rangle - a(u,v) = \langle f(W), v \rangle, \quad v \in H_0^1(X),$$

almost everywhere in $(0, T)$.
It can be shown that the bilinear form a is continuous and satisfies a Gårding inequality which implies that there exists a unique solution of this problem

Semidiscretization in time

We use the Crank-Nicolson scheme for time discretization. Let

$$M \in \mathbb{N},\ \tau = \frac{T}{M},\ t_l = l\tau,\ l = 0, \ldots, M,$$

and let us denote

$$\tilde{U}_l(x) = \tilde{U}(x, t_l), \quad f_l(x) = f(W(x, t_l)).$$

The Crank-Nicolson scheme has the form

$$\frac{\left\langle \tilde{U}_{l+1}, v \right\rangle}{\tau} - \frac{a\left(\tilde{U}_{l+1}, v\right)}{2} = \frac{\left\langle \tilde{U}_l, v \right\rangle}{\tau} + \frac{a\left(\tilde{U}_l, v\right)}{2} + \left\langle \frac{f_l + f_{l+1}}{2}, v \right\rangle$$

for $l = 0, \ldots, M-1$.

Wavelet-Galerkin method

Let Ψ be a wavelet basis for the space $L^2(X)$ such that Ψ normalized in the H^1-norm is the wavelet basis for the space $H_0^1(X)$.

Let Ψ^s be a finite-dimensional subset of Ψ with s levels of wavelets and let us denote $V^s = \operatorname{span} \Psi^s$. The Galerkin method consists in finding $\tilde{U}_{l+1}^s \in V^s$ such that

$$\frac{\left\langle \tilde{U}_{l+1}^s, v \right\rangle}{\tau} - \frac{a\left(\tilde{U}_{l+1}^s, v\right)}{2} = \frac{\left\langle \tilde{U}_l^s, v \right\rangle}{\tau} + \frac{a\left(\tilde{U}_l^s, v\right)}{2} + \left\langle \frac{f_l + f_{l+1}}{2}, v \right\rangle$$

for all $v \in V^s$.

If we set $v = \psi_\mu \in \Psi^s$ and we expand $\tilde{U}^s_{l+1}$ in a basis Ψ^s, i.e.,

$$\tilde{U}^s_{l+1} = \sum_{\psi_\lambda \in \Psi^s} u^s_\lambda \psi_\lambda,$$

then the vector of coefficients $\mathbf{u}^s = \{u^s_\lambda\}$ is the solution of the system of linear algebraic equations $\mathbf{A}^s\mathbf{u}^s = \mathbf{f}^s$, where

$$\mathbf{A}^s_{\mu,\lambda} = \frac{\langle \psi_\lambda, \psi_\mu \rangle}{\tau} - \frac{a(\psi_\lambda, \psi_\mu)}{2}$$

and

$$\mathbf{f}^s_\mu = \frac{\left\langle \tilde{U}^s_l, \psi_\mu \right\rangle}{\tau} + \frac{a\left(\tilde{U}^s_l, \psi_\mu\right)}{2} + \left\langle \frac{f_l + f_{l+1}}{2}, \psi_\mu \right\rangle.$$

It is obvious that $\mathbf{f}^s$ and $\mathbf{u}^s$ depend on the time level t_l, but for simplicity we omit the index l.

We focus on the structure of a discretization matrix. We use the Jacobi diagonal preconditioner $\mathbf{D}^s$, where the diagonal elements of $\mathbf{D}^s$ satisfy

$$\mathbf{D}^s_{\lambda,\lambda} = \sqrt{\mathbf{A}^s_{\lambda,\lambda}}.$$

We obtain the preconditioned system

$$\tilde{\mathbf{A}}^s \tilde{\mathbf{u}}^s = \tilde{\mathbf{f}}^s$$

with

$$\tilde{\mathbf{A}}^s = (\mathbf{D}^s)^{-1} \mathbf{A}^s (\mathbf{D}^s)^{-1}, \quad \tilde{\mathbf{f}}^s = (\mathbf{D}^s)^{-1} \mathbf{f}^s, \quad \tilde{\mathbf{u}}^s = \mathbf{D}^s \mathbf{u}^s.$$

We already know that the matrix arising from discretization of the differential operator $\hat{\mathcal{D}}$ is sparse and has the finger-band pattern. Hence, we focus on the integral term and we study properties of the matrix $\mathbf{C}^s$ with entries

$$\mathbf{C}^s_{\mu,\lambda} = \langle \mathcal{I}_2(\psi_\lambda), \psi_\mu \rangle, \quad \psi_\lambda, \psi_\mu \in \Psi^s.$$

For the Galerkin method with the standard spline basis the matrix $\mathbf{C}^s$ is full. However, it is known that for integral operators with some types of kernels and for wavelet bases with vanishing moments many entries of discretization matrices are small and can be thresholded. Consequently, the matrices can be approximated with matrices that are sparse matrices.

Theorem. Let Ψ be a wavelet basis with L vanishing moments, $\psi_{i,k}, \psi_{j,l} \in \Psi$ be wavelets that are generated from wavelets ψ^l, ψ^m, respectively, via translations and dilations. We denote by C_1 the maximum of the lengths of the supports of ψ^l and ψ^m and

$$C_2 = \max_{n=l,m} \int_{supp\, \psi^n} |\psi^n(x)|\, dx.$$

Let us denote $[a_{i,k}, b_{i,k}] = \Omega_{i,k} = \operatorname{supp} \psi_{i,k}$, $[a_{j,l}, b_{j,l}] = \Omega_{j,l} = \operatorname{supp} \psi_{j,l}$, $I_{i,j,k,l} = [a_{j,l} - b_{i,k}, b_{j,l} - a_{i,k}]$. If $g \in C^{2L}(I_{i,j,k,l})$, then

$$|\langle \mathcal{I}_2(\psi_{i,k}), \psi_{j,l} \rangle| \leq C_{i,j,k,l} 2^{-\left(L+\frac{1}{2}\right)(i+j)},$$

with

$$C_{i,j,k,l} = \frac{C_1^{2L} C_2^2 \lambda}{4^L (L!)^2} \max_{(x,y)\in\Omega_{i,j,k,l}} \left| g^{(2L)}(y-x) \right|.$$

Let

$$\tilde{\mathbf{C}}^s = (\mathbf{D}^s)^{-1}\,\mathbf{C}^s\,(\mathbf{D}^s)^{-1}\,.$$

Then the discretization matrix $\tilde{\mathbf{A}}^s$ is the sum of the matrix $\tilde{\mathbf{C}}^s$ and the matrix arising from discretization of the differential operator $\hat{\mathcal{D}}$.

Due the derived decay estimates, many entries of the matrix $\tilde{\mathbf{A}}^s$ are small and can be thresholded. Consequently, the matrix $\tilde{\mathbf{A}}^s$ can be represented by a sparse matrix.

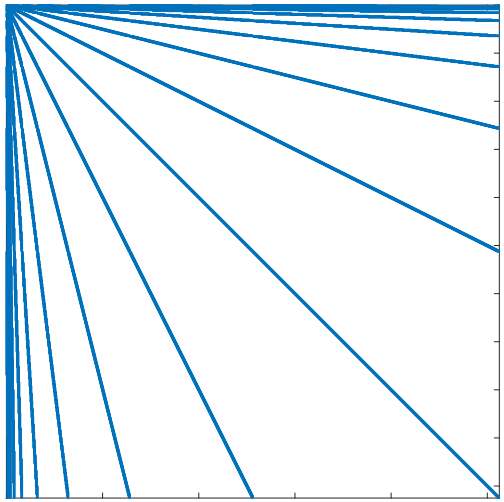


Figure: The structure of the matrix $\tilde{\mathbf{A}}^8$ that was truncated using the threshold 10^{-7} for a wavelet basis from [Černá, Finěk, 2019].

Compression strategy

We use the following strategy:

1) Choose a tolerance ϵ.
2) Compute all the entries $\tilde{\mathbf{C}}^s_{\lambda,\mu}$ for indexes $\lambda = (i,k)$ and $\mu = (j,l)$ such that $g \notin C^{2L}(I_{i,j,k,l})$.
3) Based on the decay estimate set the level $\tilde{L}$ such that $\tilde{\mathbf{C}}^s_{\lambda,\mu} < \epsilon$ for any $i+j > \tilde{L}$ and λ, μ such that $g \in C^{2L}(I_{i,j,k,l})$.
4) If $i+j \leq \tilde{L}$ then use the decay estimate to compute only entries for which it is not guaranteed that $\tilde{\mathbf{C}}^s_{\lambda,\mu} < \epsilon$.

Note that the step 4) enables us to obtain matrix $\tilde{\mathbf{C}}^s$ and thus also $\tilde{\mathbf{A}}^s$ with more zero elements, but to obtain a sparse matrix it is sufficient to use steps 1), 2), and 3), i.e., to compute entries for which $i+j \leq \tilde{L}$ and entries in regions where $g(y-x)$ is not smooth, and set to zero other entries.

The impact of the truncation on the solution of the discrete system can be described as follows. Let $\hat{\mathbf{A}}^s$ be the truncated matrix and $\hat{\mathbf{u}}^s$ be the solution of the system $\hat{\mathbf{A}}^s\hat{\mathbf{u}}^s = \tilde{\mathbf{f}}^s$. If

$$C_A = \left\| \left(\hat{\mathbf{A}}^s\right)^{-1} \cdot \left(\tilde{\mathbf{A}}^s - \hat{\mathbf{A}}^s\right) \right\| < 1,$$

then

$$\frac{\|\tilde{\mathbf{u}}^s - \hat{\mathbf{u}}^s\|}{\|\tilde{\mathbf{u}}^s\|} \leq \frac{\mathrm{cond}\tilde{\mathbf{A}}^s}{1 - C_A} \frac{\left\|\tilde{\mathbf{A}}^s - \hat{\mathbf{A}}^s\right\|}{\left\|\tilde{\mathbf{A}}^s\right\|}.$$

Morover, the matrices $\tilde{\mathbf{A}}^s$ have uniformly bounded condition numbers, i.e. $\mathrm{cond}\,\tilde{\mathbf{A}}^s < C$ with C independent on s. Hence, if we choose the threshold small enough, then $\hat{\mathbf{u}}^s$ will be close to $\tilde{\mathbf{u}}^s$.

Example 1. Merton model

We provide numerical experiments for a European vanilla option and the Merton model.

Since analytic solution is known for this model [Merton, 1976], we are able to compute the errors for the numerical solution. We use the same parameters as in [Halluin et al. 2005; Kadalbajoo et al. 2015; Kwon and Lee 2011], so that one can compare the numerical results for the methods proposed therein.

The option has the following parameters: option maturity $T = 0.25$ year, interest rate $r = 0.05$, volatility $\sigma = 0.15$, intensity $\lambda = 0.1$, and the strike price $K = 100$. The probability density function has parameters $\mu_J = -0.9$ and $\sigma_J = 0.45$.

Our domain of interest is the interval $[0, 2K]$, but because the localization error is the largest near the right endpoint of the interval, we choose $S_{max} = 4K$. To obtain $x_{min} \neq -\infty$ we have to choose $S_{min} > 0$, thus we set $S_{min} = 30$.

We use the proposed method for the computation of the value of the put option and the put-call parity to compute the value of the call option.

We use a quadratic spline wavelet basis from [Černá, 2019], where wavelets have three vanishing moments and are orthogonal if their levels i and j satisfy $|i - j| > 2$.

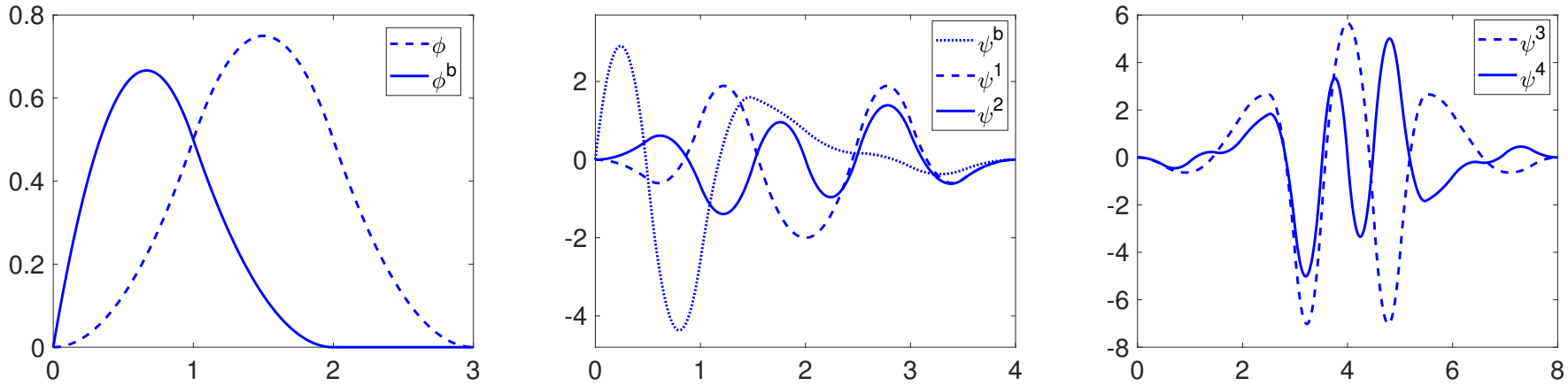


Figure: Scaling functions ϕ^b and ϕ and wavelets ψ^b, ψ^1, ψ^2, ψ^3 and ψ^4.

Due to the orthogonality property of the used basis, the discretization matrix has a simplified structure.

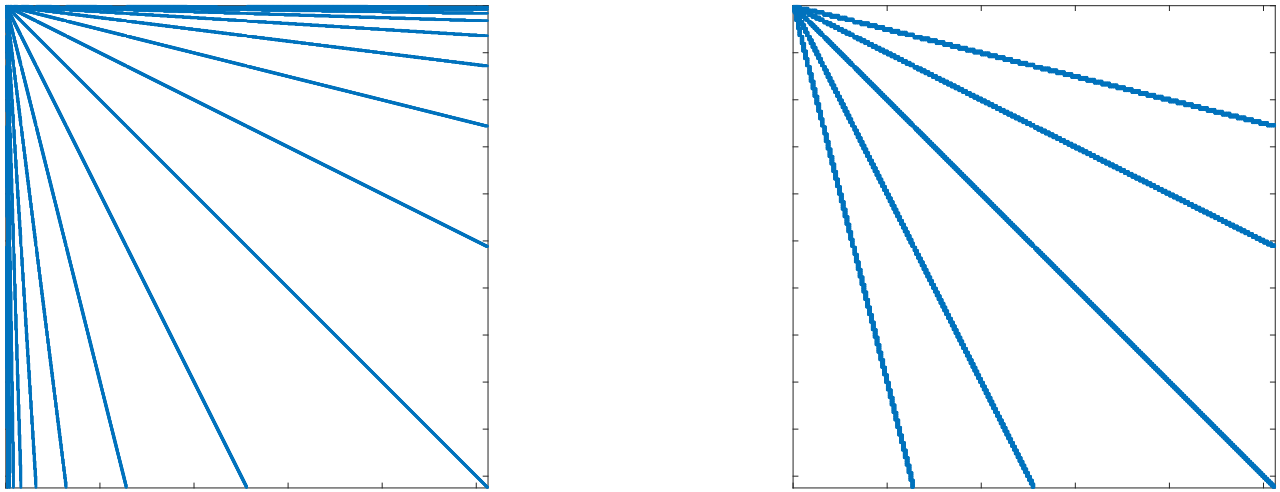


Figure: Sparse structure of the discretization matrix for wavelets from [Černá, Finěk, 2018] (left) and for the quadratic spline wavelet basis from [Černá, 2019] (right).

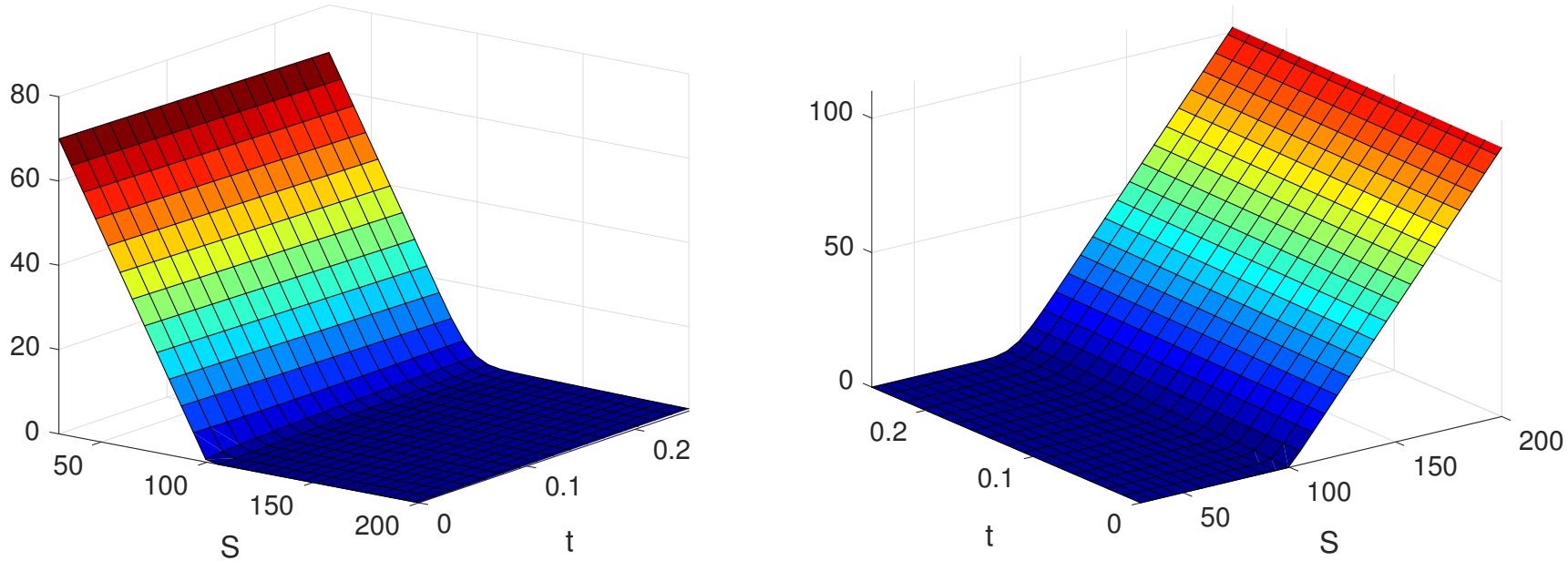


Figure: Functions representing the values of a European put (left) and call (right) option for the Merton model.

In the following table the resulting values of the options for the asset prices $S = 90$, $S = 100$, and $S = 110$, are listed.

The reference values that were computed by analytic formula from [Merton 1976] are 9.28541807 for $S = 90$, 3.14902574 for $S = 100$, and 1.40118588 for $S = 110$, for a put option.

In the table we also present the pointwise errors, i.e. the differences between the computed values and the reference values.

The wavelet-Galerkin method can be combined also with other time discretization schemes than the Crank-Nicolson scheme to obtain higher order convergence with respect to time. We improve the order of convergence with respect to τ by simple postprocessing based on Richardson extrapolation, i.e. we compute the approximate solution $U_{h,\tau}$ with the spatial step h and the time step τ and the approximate solution $U_{h,\tau/2}$ with the spatial step h and the time step $\tau/2$. Richardson extrapolation consists in computing new approximate solution

$$U^R_{h,\tau/2} = \frac{4U_{h,\tau/2} - U_{h,\tau}}{3}.$$

Then we are able to compute sufficiently accurate solution with significantly smaller number of time steps.

		Crank-Nicolson			Richardson		
S	N	M	put	error	M	put	error
90	32	6	9.284895	5.23e-4	10	9.287676	2.26e-3
	64	16	9.282265	3.15e-3	16	9.282679	2.74e-3
	128	46	9.285090	3.78e-4	27	9.285143	2.75e-4
	256	128	9.285427	8.69e-6	46	9.285433	1.51e-5
	512	363	9.285413	4.97e-6	77	9.285414	4.17e-6
	1024	1024	9.285417	6.24e-7	128	9.285418	5.39e-7

		Crank-Nicolson			Richardson		
S	N	M	put	error	M	put	error
100	32	6	3.166832	1.78e-2	10	3.165157	1.61e-2
	64	16	3.148937	8.86e-5	16	3.148590	4.36e-4
	128	46	3.149050	2.44e-5	27	3.149012	1.30e-5
	256	128	3.149038	1.20e-5	46	3.149032	6.56e-6
	512	363	3.149027	9.07e-7	77	3.149026	2.20e-7
	1024	1024	3.149026	5.64e-8	128	3.149026	1.76e-7

		Crank-Nicolson			Richardson		
S	N	M	put	error	M	put	error
110	32	6	1.389539	1.16e-2	10	1.389646	1.15e-2
	64	16	1.401664	4.78e-4	16	1.401750	5.65e-4
	128	46	1.401350	1.64e-4	27	1.401362	1.76e-4
	256	128	1.401196	1.06e-5	46	1.401198	1.20e-5
	512	363	1.401183	3.22e-6	77	1.401183	3.04e-6
	1024	1024	1.401186	3.22e-7	128	1.401186	3.00e-7

We also present the error in the norms of the space $L^2(0, 2K)$. The optimal order for the Crank-Nicolson scheme is $\mathcal{O}(\tau^2)$ and optimal order for quadratic spline approximation is $\mathcal{O}(h^3)$, where $h = 1/N$ represents the spatial step. Thus, if we decrease the spatial step to $h/2$, we decrease the time step to $\tau/\lceil\sqrt{8}\rceil$, $\lceil\cdot\rceil$ denoting the upper integer part, and we study experimental rates of convergence with respect to spatial step h computed as

$$\text{rate} = \frac{\log \text{error}\left(\frac{N}{2}, \left\lceil \frac{M}{\sqrt{a}} \right\rceil\right) - \log \text{error}\,(N, M)}{\log 2}.$$

for $a = 8$. Experimental rates of convergence for the method with Richardson extrapolation is computed for $a = 2$.

Table: Errors in the $L^2(0, 2K)$-norm and the corresponding experimental rates of convergence.

	Crank-Nicolson			Richardson		
N	M	L^2	rate	M	L^2	rate
32	6	9.91e-2	-	10	9.63e-2	-
64	16	8.00e-3	3.63	16	7.67e-3	3.65
128	46	8.80e-4	3.18	27	8.55e-4	3.17
256	128	1.07e-4	3.04	46	1.05e-4	3.03
512	363	1.33e-5	3.01	77	1.30e-5	3.01
1024	1024	1.91e-6	2.80	128	1.87e-6	2.80

Comparison of methods

We present the pointwise errors in dependence on the spatial parameter N.

WG denotes the wavelet-Galerkin method combined with the Crank-Nicolson scheme with Richardson extrapolation from this paper, FD denotes the finite difference method from [Kwon and Lee 2011], FPI denotes the method from [Halluin et al. 2005] that is based on fixed point iterations and the Crank-Nicolson scheme, and COL denotes the explicit-implicit method from [Kadalbajoo et al. 2015] that is a combination of cubic spline collocation method and backward differerence method. The number of time steps was comparable or smaller for our method than for these methods.

The spatial parameter N is significantly smaller for our method and thus significantly smaller matrices are involved in the computation.

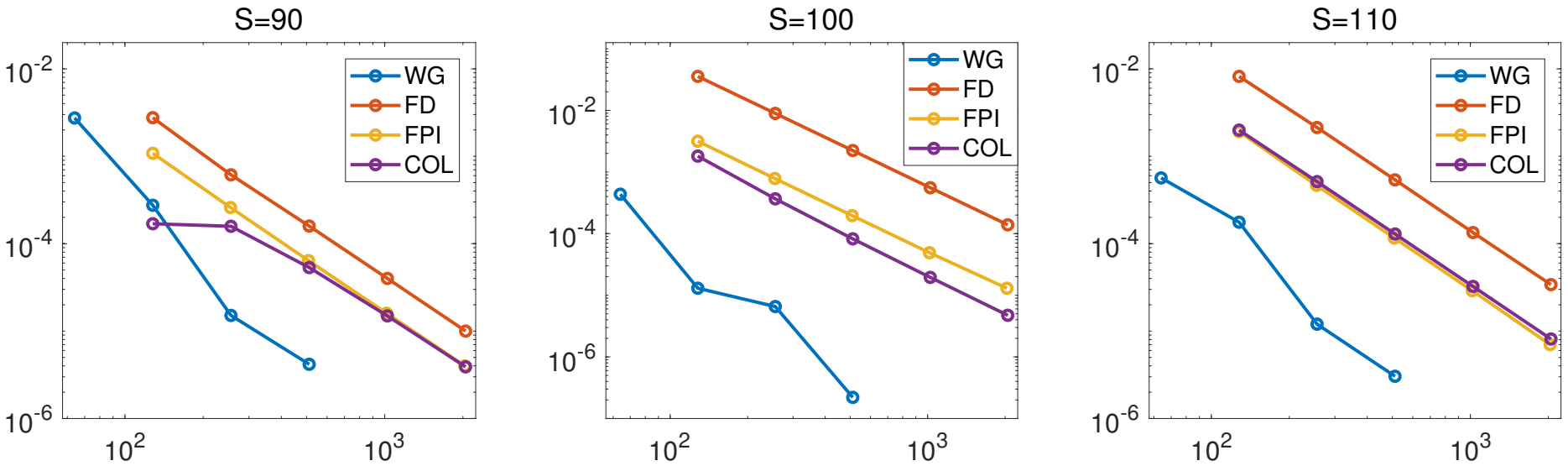


Figure: Pointwise errors for $S = 90$, $S = 100$, and $S = 110$, in dependence on the spatial parameter N for methods: WG, FD, FPI, and COL.

Condition numbers and numbers of iterations

In the following table we present the condition numbers (cond) of diagonally preconditioned matrices $\tilde{\mathbf{A}}^s$.

Furthermore, we list the number of outer and the number of inner iterations needed to resolve the resulting system of equations by the generalized minimal residual method (GMRES) with the following input parameters: restart after ten iterations, maximum number of outer iterations is 100 and the iterations stop if the relative residual is less than 10^{-12}.

Table: The condition numbers (cond) of discretization matrices, and numbers of GMRES iterations (it).

N	M	cond	it
32	6	11.5	5(1)
64	16	12.9	5(3)
128	46	14.0	5(7)
256	128	14.9	5(10)
512	363	15.7	6(2)
1024	1024	16.3	6(3)

Example 2. Kou model

We use the proposed scheme for computing a value of a European put option with parameters: $T = 0.25$, $r = 0.05$, $\sigma = 0.15$, $\lambda = 0.1$, $\eta_1 = 3.0465$, $\eta_2 = 3.0775$, $p = 0.3445$, $K = 100$.

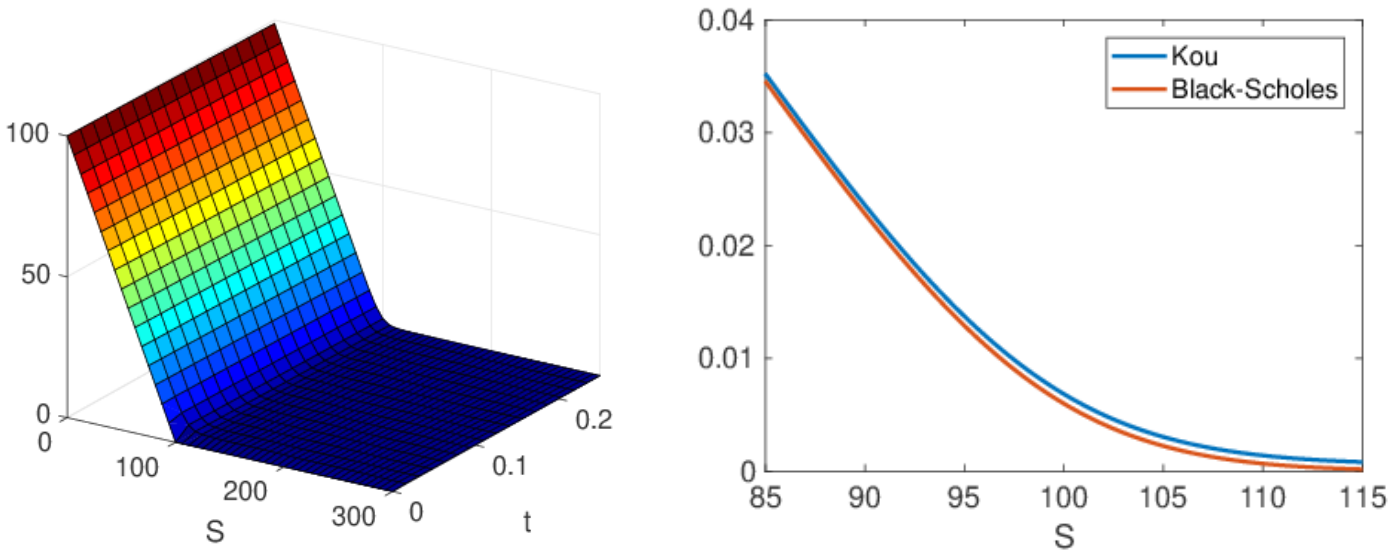
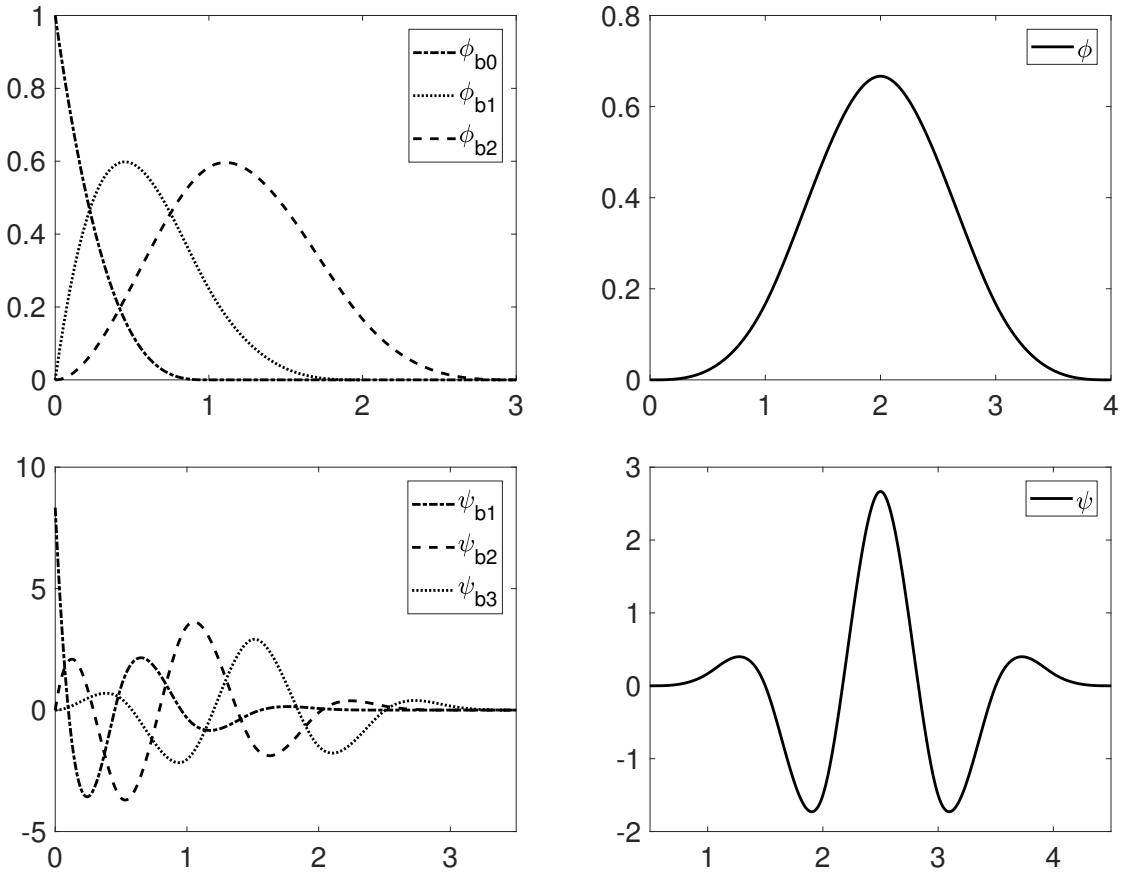


Figure: Function representing the value of a European put option for the Kou model (left), the values for the Kou and the Black-Scholes model (right).

We use a cubic spline wavelet basis with short support and four vanishing moments from [Černá, 2019].



The reference value is 9.430457 for $S = 90$, 2.731259 for $S = 100$, and 0.552363 for $S = 110$ for a put option. Reference values for call options are computed by put-call parity. In the following table we also present the pointwise errors, i.e. the differences between the computed values and the reference values, and the experimental rates of convergence computed as

$$\text{rate} = \frac{\log \text{error}\left(\frac{N+2}{2}, \frac{M}{4}\right) - \log \text{error}\left(N, M\right)}{\log 2}. \qquad (2)$$

The optimal order for the Crank-Nicolson scheme is $\mathcal{O}\left(\tau^2\right)$ and optimal order for cubic spline approximation is $\mathcal{O}\left(h^4\right)$, where $h = 1/\left(N-2\right)$ represents the spatial step. It seems that the errors presented in the table correspond to the optimal order $\mathcal{O}\left(h^4 + \tau^2\right)$.

S	N	M	put	call	error	rate
90	18	1	9.558719	0.800939	1.28e-1	
	34	4	9.465419	0.707639	3.50e-2	1.87
	66	16	9.429905	0.672125	5.52e-4	5.99
	130	64	9.430436	0.672656	2.21e-5	4.64
	258	256	9.430464	0.672684	6.88e-6	1.68
100	18	1	2.949505	4.191725	2.18e-1	
	34	4	2.707230	3.949449	2.40e-2	3.18
	66	16	2.729640	3.971860	1.62e-3	3.89
	130	64	2.731205	3.973425	5.31e-5	4.93
	258	256	2.731261	3.973481	1.58e-6	5.07

S	N	M	put	call	error	rate
110	18	1	0.264414	11.506634	2.88e-1	
	34	4	0.550486	11.792706	1.88e-3	7.26
	66	16	0.551193	11.793413	1.17e-3	0.68
	130	64	0.552399	11.794619	3.53e-5	5.05
	258	256	0.552369	11.794588	5.78e-6	2.61

Table: Errors for values of European vanilla options for various methods.

Kwon, Lee [8]			d'Halluin et al. [4]			Kadalbajoo et al. [6]		
N	M	error	N	M	error	N	M	error
128	25	3.47e-2	128	34	3.51e-3	128	12	1.96e-3
256	50	8.72e-3	255	65	1.00e-3	256	24	3.98e-4
512	100	2.17e-3	509	132	3.72e-4	512	48	8.96e-5
1024	200	5.42e-4	1017	266	1.57e-4	1024	96	2.15e-5
2048	400	1.36e-4	2033	533	7.20e-5	2048	192	5.60e-6

basis	N	M	nnz	cond	it
short4	34	4	891	36	7(6)
	66	16	2105	38	7(8)
	130	64	4716	40	7(10)
	258	256	10042	40	7(9)
	514	1024	21183	41	7(6)
bior4.6	34	4	1061	2.0e3	83(5)
	66	16	2895	1.4e5	100(10)
	130	64	7474	6.6e6	100(10)
	258	256	19024	3.2e8	100(10)
B-spline	34	4	1156	17	5(5)
	66	16	4333	18	5(9)
	130	64	16616	18	5(7)
	258	256	64111	18	5(6)
	514	1024	244518	18	5(5)

Conclusions

- We used the Crank-Nicolson scheme combined with wavelet-Galerkin method for option pricing under jump-diffusion model.
- We derived estimates for entries of discretization matrices and proposed compression strategy which leads to sparse matrices.
- The high order convergence was achived when the Crank-Nicolson scheme was combined with Richardson iterations and quadratic or cubic spline wavelets were used.
- The structure of the discretization matrix was simplified using wavelets satisfying special orthogonality conditions.
- The discretization matrices were well-conditioned and the number of iterations was small.
- The small number of parameters was used to represent the solution with desired accuracy.

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