Laplaceova transformace

Příklad 1. Pomocí Laplaceovy transformace řešte na intervalu $(0,\infty)$ diferenciální rovnice:

a) $y'' + 4 = \cos 2t,\ y(0) = 1,\ y'(0) = 0$ $\left[\frac{5}{4} - \frac{\cos 2t}{4} - 2t^2\right]$

b) $y' + 2y = 3,\ y(0) = 0$ $\left[\frac{3}{2} - \frac{3}{2}e^{-2t}\right]$

c) $y'' + y = \sin t \cos t,\ y(0) = 0,\ y'(0) = 0$ $\left[\frac{\sin t}{3} - \frac{\sin 2t}{6}\right]$

d) $y'' + 2y' = \sin\left(t + \frac{\pi}{2}\right),\ y(0) = -1,\ y'(0) = 2$ $\left[-\frac{4e^{-2t}}{5} - \frac{\cos t}{5} + \frac{2\sin t}{5}\right]$

e) $y'' + 2y' + 5y = e^{-t},\ y(0) = 0,\ y'(0) = 2$ $\left[e^{-t}\sin 2t + \frac{e^{-t}}{4} - \frac{e^{-t}\cos 2t}{4}\right]$

f) $y''' - 3y'' + 3y' - y = e^t,\ y(0) = 1,\ y'(0) = 1,\ y''(0) = 1$ $\left[e^t + \frac{t^3e^t}{6}\right]$

g) $y'' + y - 1 = -2\sin^2\frac{t}{2},\ y(0) = -1,\ y'(0) = 1$ $\left[\frac{1}{2}t\sin t - \cos t + \sin t\right]$

Příklad 2. Pomocí Laplaceovy transformace řešte diferenciální rovnice:

a) $y'' - 4y' + 4y = e^{2t},\ y(0) = 0,\ y(1) = e^2$ $\left[\frac{t^2e^{2t}}{2} + \frac{te^{2t}}{2}\right]$

b) $y'' + y = 3\sin t\cos t,\ y(0) = 1,\ y\left(\frac{\pi}{2}\right) = 2$ $\left[2\sin t - \frac{\sin 2t}{2} + \cos t\right]$

c) $y'' + 2y' + y = e^{-2t},\ y(0) = 0,\ y(1) = 5$ $\left[e^{-2t} + (t-1)e^{-t} + 5te^{1-t} - te^{-1-t}\right]$

Příklad 3. Určete obraz Laplaceovy transformace funkce f a nakreslete graf této funkce.

a) $f(t) = 0$ pro $t \in [0,1)$, $f(t) = 2$ pro $t \geq 1$ $\left[\frac{2e^{-p}}{p}\right]$

b) $f(t) = 3$ pro $t \in [1,2)$, $f(t) = 0$ pro $t \notin [1,2)$ $\left[\frac{3e^{-p}}{p} - \frac{3e^{-2p}}{p}\right]$

c) $f(t) = 0$ pro $t \in [0,\pi)$, $f(t) = \cos t$ pro $t \geq \pi$ $\left[-\frac{p}{p^2+1}e^{-\pi p}\right]$

d) $f(t) = 0$ pro $t \in [0,2)$, $f(t) = t - 2$ pro $t \geq 2$ $\left[\frac{e^{-2p}}{p^2}\right]$

e) $f(t) = 1 - t$ pro $t \in [0,1)$, $f(t) = 0$ pro $t \geq 1$ $\left[\frac{1}{p} - \frac{1}{p^2} + \frac{e^{-p}}{p^2}\right]$

f) $f(t) = 1$ pro $t \in [0,1)$, $f(t) = 2$ pro $t \geq 1$ $\left[\frac{1}{p} + \frac{e^{-p}}{p}\right]$

g) $f(t) = 0$ pro $t \in [0,1)$, $f(t) = (t-1)^3$ pro $t \geq 1$ $\left[\frac{6e^{-p}}{p^4}\right]$

h) $f(t) = \sin t$ pro $t \in [0,\pi)$, $f(t) = 0$ pro $t \geq \pi$ $\left[\frac{1}{p^2+1} + \frac{e^{-\pi p}}{p^2+1}\right]$

Příklad 4. Určete funkci f, pro kterou je obraz Laplaceovy transformace funkce F.

a) $F(p) = \frac{p}{p^2+1} e^{-2p}$ $[\cos(t-2) H(t-2)]$

b) $F(p) = \frac{e^{-p} - e^{-3p}}{p}$ $[H(t-1) - H(t-3)]$

c) $F(p) = \frac{e^{-2p}}{p+2}$ $\left[e^{4-2t} H(t-2)\right]$

d) $F(p) = \frac{e^{-5p}}{(p-1)^2 + 9}$ $\left[\frac{e^{t-5} \sin(3t-15)}{3} H(t-5)\right]$

e) $F(p) = \frac{e^{-6p}}{p^2}$ $[(t-6) H(t-6)]$

f) $F(p) = \frac{e^{-2p}}{p-1}$ $\left[e^{t-2} H(t-2)\right]$

g) $F(p) = \frac{1 - e^{-p}}{p}$ $[1 - H(t-1)]$

h) $F(p) = \frac{e^{-3p}}{p^4}$ $\left[\frac{(t-3)^3 H(t-3)}{6}\right]$

Příklad 5. Pomocí Laplaceovy transformace řešte na intervalu $(0, \infty)$ diferenciální rovnice:

a) $y'' + y' = f,\ y(0) = 0,\ y'(0) = 0,\ f(t) = t - 1$ pro $t \geq 1, f(t) = 0$ pro $t < 1$.

$$\left[\left(\frac{t^2}{2} - 2t + \frac{5}{2} - e^{1-t}\right) H(t-1)\right]$$

b) $y'' - 2y' + y = f,\ y(0) = 0,\ y'(0) = 1,\ f(t) = 3$ pro $t \in [0,1), f(t) = 0$ pro $t \in [1, \infty)$.

$$\left[4te^t + 3 - 3e^t - 3H(t-1) - 3te^{t-1} H(t-1) + 6e^{t-1} H(t-1)\right]$$